IMPROVED BREAKDOWN CRITERION FOR EINSTEIN VACUUM EQUATIONS IN CMC GAUGE

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1. Introduction

Let $(\mathbf{M}, \mathbf{g})$ be a (3+1)-dimensional vacuum globally hyperbolic space-time, i.e. $\mathbf{g}$ is a Lorentz metric of signature $(-,+,+,+)$ satisfying the Einstein vacuum equations

$$\mathbf{Ric}(\mathbf{g}) = 0$$

and every causal curve intersects a Cauchy surface at precisely one point. If $(\mathbf{M}, \mathbf{g})$ has a compact, constant mean curvature (CMC) Cauchy surface Σ_0 with mean curvature $t_0 < 0$, then there exists a foliation of a neighborhood of Σ_0 by compact CMC surfaces, and the mean curvature varies monotonically from slice to slice. The CMC conjecture states that there is a foliation in $\mathbf{M}$ of CMC Cauchy surfaces with mean curvatures taking on all allowable values, i.e. the mean curvatures take all values in $(-\infty, 0)$ if Σ_0 is of Yamabe type -1 or 0, while the mean curvatures take on all values in $(-\infty, \infty)$ if Σ_0 is of Yamabe type $+1$. Certain progress has been made ([3]), the CMC conjecture however remains open. One of the important step to attack the CMC conjecture is to provide a reasonable breakdown criterion to detect what may happen when the CMC foliation can not be extended.

In order to set up the framework, in this paper we assume that $\mathcal{M}_*$ is a part of the space-time $(\mathbf{M}, \mathbf{g})$ foliated by CMC hypersurfaces Σ_t with mean curvature t satisfying $t_0 \le t < t_*$ for some $t_0 < t_* < 0$. We shall refer to $\Sigma_0 := \Sigma_{t_0}$ as the initial slice. Thus, $\mathcal{M}_* = \bigcup_{t\in[t_0,t_*)} \Sigma_t$ with $t_* < 0$ and there is a time function t defined on $\mathcal{M}_*$, monotonically increasing toward the future, such that each Σ_t is a level hypersurface of t with the lapse function n and the second fundamental form k defined by

$$n := (-\mathbf{g}(\mathbf{D}t, \mathbf{D}t))^{1/2} \quad \text{and} \quad k(X, Y) := -\mathbf{g}(\mathbf{D}_X \mathbf{T}, Y),$$

where $\mathbf{T}$ denotes the future directed unit normal to Σ_t, $\mathbf{D}$ denotes the space-time covariant differentiation associated with $\mathbf{g}$, and X, Y are vector fields tangent to Σ_t. Let g be the induced Riemannian metric on Σ_t and let ∇ be the corresponding covariant differentiation. For any coordinate chart $\mathcal{O} \subset \Sigma_0$ with coordinates $x = (x^1, x^2, x^3)$, let $x^0 = t, x^1, x^2, x^3$ be the transported coordinates on $[t_0, t_*) \times \mathcal{O}$ obtained by following the integral curves of $\mathbf{T}$. Under these coordinates the metric $\mathbf{g}$ takes the form

$$\mathbf{g} = -n^2 dt^2 + g_{ij} dx^i dx^j. \tag{1.1}$$

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Moreover, relative to these coordinates t, x^1, x^2, x^3 there hold the evolution equations

$$\partial_t g_{ij} = -2nk_{ij}, \tag{1.2}$$

$$\partial_t k_{ij} = -\nabla_i\nabla_j n + n(R_{ij} + \mathrm{Tr}k\, k_{ij} - 2k_{ia}k_j^a) \tag{1.3}$$

and the constraint equations

$$R - |k|^2 + (\mathrm{Tr}k)^2 = 0, \tag{1.4}$$

$$\nabla^j k_{ji} - \nabla_i \mathrm{Tr}k = 0 \tag{1.5}$$

on each Σ_t, where R_{ij} and R denote the Ricci curvature and the scalar curvature of the induced metric g on Σ_t, and $\mathrm{Tr}k$ denotes the trace of k, i.e. $\mathrm{Tr}k = g^{ij}k_{ij}$. Since $\mathrm{Tr}k = t$ on Σ_t, it follows from the above equations that

$$\mathrm{div}k = 0 \tag{1.6}$$

and

$$-\Delta n + |k|^2 n = 1 \tag{1.7}$$

on each Σ_t.

The first important breakdown criterion was given by M. Anderson in [2], who showed that when a breakdown occurs at t_* there holds

$$\limsup_{t\to t_*^-} \|\mathbf{R}\|_{L^\infty(\Sigma_t)} = \infty,$$

where $\mathbf{R}$ denotes the Riemannian curvature tensor of the space-time $(\mathbf{M}, \mathbf{g})$. Here the pointwise norm $|\mathbf{R}|$ is defined with respect to the Riemannian metric $\mathbf{g_T}$ on $\mathbf{M}$, where $\mathbf{g_T}$ is defined as follows: for any $X, Y \in T\mathcal{M}_*$ write

$$X = X^0\mathbf{T} + \underline{X} \quad \text{and} \quad Y = Y^0\mathbf{T} + \underline{Y}$$

with $\underline{X}, \underline{Y} \in T\Sigma_t$, then

$$\mathbf{g_T}(X, Y) = X^0Y^0 + g(\underline{X}, \underline{Y}).$$

The result of Anderson implies that if

$$\sup_{t\in[t_0,t_*)} \|\mathbf{R}\|_{L^\infty(\Sigma_t)} = \Lambda_0 < \infty \tag{1.8}$$

for all $t_* < 0$, then the CMC foliation exists for all values in $[t_0, 0)$.

Recently, Klainerman and Rodnianski [12] provided a new breakdown criterion which shows that if a breakdown happens at $t_* < 0$ then

$$\limsup_{t\to t_*^-} \left(\|k\|_{L^\infty(\Sigma_t)} + \|\nabla \log n\|_{L^\infty(\Sigma_t)}\right) = \infty,$$

or, in other words, the CMC foliation can be extended beyond any value $t_* < 0$ for which

$$\sup_{t\in[t_0,t_*)} \left(\|k\|_{L^\infty(\Sigma_t)} + \|\nabla \log n\|_{L^\infty(\Sigma_t)}\right) = \Lambda_0 < \infty. \tag{1.9}$$

In contrast to the breakdown criterion of Anderson, the condition (1.9) of Klainerman and Rodnianski is formally weaker as it refers only to the second fundamental form k and the lapse function n which requires one degree less of differentiability. Moreover, by purely elliptic estimates, one can see that (1.8) implies immediately (1.9), since the boundedness of $\|\mathbf{R}\|_{L^\infty}$ exhausts all the dynamical degrees of freedom of the equations. Therefore, the result in [12] is a significant improvement.

We remark that the result of Klainerman and Rodnianski can not be established by purely elliptic estimates. Instead, the proof relies heavily on the tools from the theory of hyperbolic equations. The analogous result has been extended to non-vacuum space-time in [15].

If we consider the Einstein equation expressed relative to the wave coordinates, by energy estimates one can see that the breakdown does not occur unless

$$\int_{t_0}^{t_*} \|\partial \mathbf{g}\|_{L^\infty} dt = \infty. \tag{1.10}$$

This condition however is not geometric since it depends on the choice of a full coordinate system. Observe that the components of the second fundamental form k and ∇n can be viewed as part of the components of $\partial\mathbf{g}$. It is natural to ask if we have an integral form of breakdown criterion involving k and n only. The first main result of the present paper confirms this and provides a geometric counterpart of (1.10), which can be viewed as an improved version of the breakdown criterion of Klainerman and Rodnianski.

Theorem 1.1 (Main theorem I)**.** [1] *Let $(\mathcal{M}_*, \boldsymbol{g})$ be a globally hyperbolic development of Σ_0 foliated by the CMC level hypersurfaces of a time function $t < 0$. Then the space-time together with the foliation Σ_t can be extended beyond any value $t_* < 0$ for which,*

$$\int_{t_0}^{t_*} \left(\|k\|_{L^\infty(\Sigma_t)} + \|\nabla \log n\|_{L^\infty(\Sigma_t)}\right) dt = \mathcal{K}_0 < \infty. \tag{1.11}$$

Let us fix the convention for the deformation tensor of $\mathbf{T}$, expressed relative to an orthonormal frame $\{e_0 = \mathbf{T}, e_1, e_2, e_3\}$, as follows,

$$\pi_{\alpha\beta} = -\mathbf{g}(\mathbf{D}_{e_\alpha}\mathbf{T}, e_\beta), \text{ with } \alpha, \beta = 0, 1, 2, 3.$$

It is easy to check

$$\pi_{00} = 0,\ \pi_{0i} = -\nabla_i \log n,\ \pi_{i0} = 0,\ \pi_{ij} = k_{ij}, \text{ with } i, j = 1, 2, 3.$$

Consequently, the condition (1.9) can be formulated as

$$\sup_{t\in[t_0,t_*)} \|\pi\|_{L^\infty(\Sigma_t)} = \Lambda_0 < \infty,$$

while the weaker condition (1.11) can be formulated as

$$\|\pi\|_{L^1_t L^\infty_x(\mathcal{M}_*)} := \int_{t_0}^{t_*} \|\pi\|_{L^\infty(\Sigma_t)} dt = \mathcal{K}_0 < \infty. \tag{\textbf{A1}}$$

We basically follow the framework in [12] to prove Theorem 1.1; however, a sequence of difficulties occur due to the weaker condition (1.11). In order to continue the foliation, according to the local existence theorem given in [5, Theorem 10.2.1], one must establish a global uniform bound for the curvature tensor $\mathbf{R}$ and L^2-bounds for its first two covariant derivatives. Since $(\mathbf{M}, \mathbf{g})$ is a vacuum space-time, by virtue of the Bianchi identity $\mathbf{R}$ verifies a wave equation of the form

$$\Box_{\mathbf{g}}\mathbf{R} = \mathbf{R} \star \mathbf{R}, \tag{1.12}$$

where $\Box$ denotes the covariant wave operator $\Box = \mathbf{D}^\alpha \mathbf{D}_\alpha$. Based on higher energy estimates it is standard to show that the L^2 bounds for $\mathbf{DR}$ and $\mathbf{D}^2\mathbf{R}$ can be

[1] Our method applies equally well to the case that Σ_t are asymptotically flat and maximal, i.e $\mathrm{Tr}k = 0$ and can also be extended to Einstein space-time with matter.

bounded in terms of the L^∞ norm of $\mathbf{R}$. Thus, the derivation of the L^∞ bound of $\mathbf{R}$ is a crucial step. In order to achieve this goal, Klainerman and Rodnianski [10] succeeded in representing $\mathbf{R}(p)$, for each $p \in \mathcal{M}_*$, by a Kirchoff-Sobolev formula of the form

$$\mathbf{R}(p) = -\int_{\mathcal{N}^-(p,\tau)} \mathbf{A}\cdot(\mathbf{R}\star\mathbf{R}) + \text{other terms}$$

where $\mathbf{A}$ is a 4-covariant tensor defined as a solution of a transport equation along $\mathcal{N}^-(p,\tau)$ with appropriate initial data at the vertex p, $\mathcal{N}^-(p,\tau)$ denotes the portion of the null boundary $\mathcal{N}^-(p)$ in the time interval $[t(p)-\tau, t(p)]$. The past null cone $\mathcal{N}^-(p)$ is in general an achronal Lipschitz hypersurface ruled by the set of past null geodesics from p. In order to derive all necessary estimates, one must show that $\mathcal{N}^-(p)$ remains a smooth hypersurface in the time slab $[t(p)-\tau, t(p))$ for some universal constant $\tau > 0$. Therefore, it is necessary to provide a uniform lower bound for the past null radius of injectivity at all $p \in \mathcal{M}_*$.

Let us recall briefly the definition of the past null radius of injectivity at p, one may consult [11] for more details. We parametrize the set of past null vectors in $T_p\mathbf{M}$ in terms of $\omega \in \mathbb{S}^2$, the standard sphere in $\mathbb{R}^3$. Then, for each $\omega \in \mathbb{S}^2$, let l_ω be the null vector in $T_p\mathbf{M}$ normalized with respect to the future, unit, time-like vector $\mathbf{T}_p$ by

$$\mathbf{g}(l_\omega, \mathbf{T}_p) = 1$$

and let $\Gamma_\omega(s)$ be the past null geodesic with initial data $\Gamma_\omega(0) = p$ and $\frac{d}{ds}\Gamma_\omega(0) = l_\omega$. We define the null vector field L on $\mathcal{N}^-(p)$ by

$$L(\Gamma_\omega(s)) = \frac{d}{ds}\Gamma_\omega(s)$$

which may only be smooth almost everywhere on $\mathcal{N}^-(p)$ and can be multi-valued on a set of exceptional points. We can choose the parameter s with $s(p) = 0$ so that

$$\mathbf{D}_L L = 0 \quad \text{and} \quad L(s) = 1.$$

This s is called the affine parameter.

The past null radius of injectivity $i_*(p)$ at p is then defined to be the supremum over all the values $s_0 > 0$ for which the exponential map

$$\mathfrak{g}_p : (s,\omega) \to \Gamma_\omega(s)$$

is a global diffeomorphism from $(0, s_0)\times\mathbb{S}^2$ to its image in $\mathcal{N}^-(p)$. It is known that $i_*(p) > 0$ for each p, $\mathcal{N}^-(p)$ is smooth within the null radius of injectivity, and

$$i_*(p) = \min\{s_*(p), l_*(p)\},$$

where $s_*(p)$, the past null radius of conjugacy at p, is defined to be the supremum over all values $s_0 > 0$ such that the exponential map $\mathfrak{g}_p$ is a local diffeomorphism from $(0, s_0)\times\mathbb{S}^2$ to its image in $\mathcal{N}^-(p)$, and $l_*(p)$, the past cut locus radius at p, is defined to be the smallest value of s_0 for which there exist two distinct null geodesics Γ_1 and Γ_2 from p with $\Gamma_1(s_0) = \Gamma_2(s_0)$. Thus, for a past null geodesic Γ_ω from p, a point $q = \Gamma_\omega(s_*)$ is called a conjugate point of p if $\mathfrak{g}_p$ is singular at (s_*, ω), while it is called a null cut point of p if $\mathfrak{g}_p$ is nonsingular at (s_*, ω) and through q there exists another null geodesic emanating from p.

Since we are working on the CMC foliation, it is convenient to introduce the past null radius of injectivity $i_*(p,t)$ at each p with respect to the global time function

t. We define $i_*(p,t)$ to be the supremum over all the values $\tau > 0$ for which the exponential map

$$\mathcal{G}_p : (t,\omega) \to \Gamma_\omega(s(t)) \tag{1.13}$$

is a global diffeomorphism from $(t(p)-\tau, t(p)) \times \mathbb{S}^2$ to its image in $\mathcal{N}^-(p)$. We remark that s is a function not only depending on t but also on ω, we suppress ω just for convenience. It is known that

$$i_*(p,t) = \min\{s_*(p,t), l_*(p,t)\},$$

where $s_*(p,t)$ is defined to be the supremum over all values $\tau > 0$ such that the map $\mathcal{G}_p$ is a local diffeomorphism from $(t(p)-\tau, t(p)) \times \mathbb{S}^2$ to its image, and $l_*(p,t)$ is defined to be the smallest value of $\tau > 0$ for which there exist two distinct null geodesics $\Gamma_1(s(t))$ and $\Gamma_2(s(t))$ from p which intersect at a point with $t = t(p) - \tau$.

In [11] Klainerman and Rodnianski provided a uniform lower bound on the null radius of injectivity under the assumption (1.9). In order to complete the proof of Theorem 1.1, one must provide a uniform lower bound on the null radius of injectivity under the weaker condition (1.11). This is contained in the second main result of the present paper.

Theorem 1.2 (Main theorem II)**.** *Assume that $\mathcal{M}_*$ is a globally hyperbolic development of Σ_0 verifying the condition (1.11). Then for all $p \in \mathcal{M}_*$ there holds*

$$i_*(p,t) > \min\{\delta_*, t(p) - t_0\}, \tag{1.14}$$

where $\delta_* > 0$ is a constant depending only on Q_0, $\mathcal{K}_0$, $|\Sigma_0|$ and t_*. [2]

In order to prove this result, it is useful to review the essential steps in the work of Klainerman and Rodnianski in [11]. The first step is to show that

$$s_*(p,t) > \min\{l_*(p,t), \delta_*\} \tag{1.15}$$

for some universal constant[3] $\delta_* > 0$. This can be achieved by showing that

$$\sup_{\mathcal{N}^-(p,\tau)} \left| \mathrm{tr}\chi - \frac{2}{s(t)} \right| \le C \tag{1.16}$$

with $\tau := \min\{l_*(p,t), \delta_*\}$, where χ is the null second fundamental form $\chi_{AB} = \mathbf{g}(\mathbf{D}_A L, e_B)$ of the 2-dimensional space-like surface $S_t := \mathcal{N}^-(p) \cap \Sigma_t$ with $(e_A)_{A=1,2}$ being a frame field tangent to S_t. The analog has been carried out in [7, 8, 9, 13] for geodesic foliations under the boundedness assumption of the curvature flux. In order to adapt those arguments to prove (1.16) for the time foliations, one needs to show that $t(p) - t$ and s are comparable and the geodesic curvature flux (see [11]) is bounded, both of which rely on the relation

$$|a - 1| \le \frac{1}{2} \quad \text{on } \mathcal{N}^-(p,\tau), \tag{1.17}$$

where a, the null lapse function, is defined by $a^{-1} := \mathbf{g}(\mathbf{T}, L)$ with $a(p) = 1$. Note that along a null geodesic

$$\frac{dt}{ds} = -(an)^{-1}, \quad \frac{da}{ds} = \nu, \quad \nu := k_{NN} - \nabla_N \log n,$$

[2] Q_0 denotes the Bel-Robinson energy on the initial slice Σ_0 which will be defined in Section 2.

[3] A universal constant always means a constant depending only on Q_0, $\mathcal{K}_0$, $|\Sigma_0|$, t_* and the number $I_0 > 0$ such that $I_0^{-1} \le (g_{ij}) \le I_0$ on the initial slice Σ_0. Throughout this paper C always denotes a universal constant.

where N is the unit inward normal of S_t in Σ_t. If (1.9) is satisfied, one can see that (1.17) holds for $t(p) - \delta_* \le t \le t(p)$ for some universal $\delta_* > 0$, and consequently s and $t(p) - t$ are comparable. However, under the weaker condition (1.11) only, it is highly nontrivial to obtain (1.17). We observe that (1.17) can be achieved by establishing

$$\|\nu\|^2_{L^\infty_\omega L^2_t(\mathcal{N}^-(p,\tau))} = \sup_{\omega\in\mathbb{S}^2} \int_{\Gamma_\omega} an|\nu|^2 dt \le C \tag{1.18}$$

where Γ_ω is the portion of a past null geodesic initiating from p contained in $\mathcal{N}^-(p,\tau)$, for some universal constant $\delta_* > 0$. How to obtain such an estimate on ν is the first difficulty we encounter. The idea to derive the trace estimate (1.18) is to employ the techniques in the proof of the sharp trace inequality in [7, 8, 9]. Under the assumption (1.11) only, suppose the sharp trace inequality holds true on null cone in time foliation, in order to prove (1.18), schematically, we need to prove

i) there holds for $\nabla\!\!\!/ \nu$ the decomposition

$$\nabla\!\!\!/ \nu = \nabla_L P + Q \tag{1.19}$$

with P and Q appropriate S_t tangent tensors. [4]

ii) there holds

$$\|\nabla\!\!\!/ (\nu, P)\|_{L^2(\mathcal{N}^-(p,\tau))} + \|\nabla_L(\nu, P)\|_{L^2(\mathcal{N}^-(p,\tau))} \le C. \tag{1.20}$$

The decomposition of the form (1.19) will be derived in [14]. To prove the sharp trace inequality in time foliation and to control P and Q must be coupled with the proof of a series of estimates for the Ricci coefficients on null hypersurface $\mathcal{N}^-(p,\tau)$ including (1.16) by a delicate bootstrap argument. Hence, under the condition (1.11) only, (1.16), (1.17) and (1.18) should be proved simultaneously. The proof is rather involved and close to the spirit of the works [7, 8, 9, 13]. We will present it in [14] with full details.

Now we simply consider how to obtain the estimate for ν in (1.20). The estimate for $\nabla_N \log n$ of the form (1.20) can be obtained by elliptic estimates and trace inequality. By elliptic estimate, in view of

$$\operatorname{div} k = 0, \ \operatorname{curl} k = H, \tag{1.21}$$

where H denotes the magnetic part of $\mathbf{R}$, we can only derive $\|k\|_{H^1_x(\Sigma)} \le C$, which, by classic trace theorem, loses $1/2$ derivative if restricted to null cone. However, (1.20) requires the L^2 control of one derivative of k_{NN} on null cones. Hence, we must adopt a different approach, which significantly surpasses the one via elliptic estimate and trace inequality. This inspires us to use the tensorial wave equation for k, which symbolically is given by

$$\Box k = k\cdot Ric + n^{-2}\nabla^2\dot{n} + \pi\cdot\nabla k - n^{-3}\dot{n}\nabla^2 n + \pi\cdot\pi\cdot\pi + k\cdot\nabla^2 n - n^{-1}k. \tag{1.22}$$

We then prove by energy method, the k-flux satisfies

$$\|\nabla\!\!\!/ k\|_{L^2(\mathcal{N}^-(p,\tau))} + \|\nabla_L k\|_{L^2(\mathcal{N}^-(p,\tau))} \le C, \tag{1.23}$$

which schematically gives the desired control on k_{NN}.

The next step is to find a system of good local space-time coordinates under which $\mathbf{g}$ is comparable with the Minkowski metric. More precisely, for a sufficiently small constant $\epsilon > 0$, one needs to show that there exists a constant $\delta_* > 0$,

[4] $\nabla\!\!\!/$ denotes the connection with respect to the induced metric γ on S_t.

depending only on ϵ and some universal constants, for which each geodesic ball $B_{\delta_*}(p)$ with $p \in \Sigma_t$ admits local coordinates $x = (x^1, x^2, x^3)$ such that under the corresponding transport coordinates $x^0 = t, x^1, x^2, x^3$ the metric $\mathbf{g}$ has the expression (1.1) with

$$|n - n(p)| \le \epsilon \qquad \text{and} \qquad |g_{ij} - \delta_{ij}| \le \epsilon \tag{1.24}$$

on $B_{\delta_*}(p) \times [t(p) - \delta_*, t(p)]$. The existence of such local coordinates together with (1.16) will enable us to show that $\mathcal{N}^-(p, \delta_*)$ is close to the flat cone and consequently $l_*(p,t) \ge \delta_*$.

The part on n in (1.24) can be established by elliptic estimates on n and $\partial_t n$. The derivation of the result for g under the weaker condition (1.11), however, presents one of the core difficulties, which invokes new methods and a second application of (1.22).

By the Bel-Robinson energy bound $\mathcal{Q}(t) \le C$ and a result of Anderson [1], one can control the lower bound of harmonic radius on Σ_t, such that with the coordinates $x = (x^1, x^2, x^3)$ on $B_{\delta_*}(p) \subset \Sigma_t$,

$$|g_{ij}(x, t(p)) - \delta_{ij}| \le \frac{1}{2}\epsilon.$$

The challenge is to control time evolution of g. Using (1.2), one has [5]

$$|g_{ij}(x, t(p)) - g_{ij}(x,t)| \lesssim \int_t^{t(p)} |k(x,t')| dt'. \tag{1.25}$$

If (1.9) holds, or more generally, if

$$\int_{t_0}^{t_*} \|k(t')\|^q_{L^\infty(\Sigma_{t'})} dt' \le \Lambda_0 < \infty$$

for some $q > 1$, then with δ_* sufficiently small

$$|g_{ij}(x, t(p)) - g_{ij}(x,t)| \le \Lambda_0^{1/q} (t(p) - t)^{1 - 1/q} < \frac{1}{2}\epsilon. \tag{1.26}$$

The above argument fails if k verifies (1.11) only. Under the assumption (1.11), our strategy is to prove directly the integral on the right of (1.25) can be small, i.e.

$$\int_t^{t(p)} |k(x,t')| dt' < \frac{1}{2}\epsilon, \qquad \forall x \in \Sigma$$

by establishing

$$\sup_{x \in \Sigma} \int_t^{t(p)} |k(x,t')|^2 dt' \le C, \tag{1.27}$$

since

$$|g_{ij}(x, t(p)) - g_{ij}(x,t)| \lesssim \left(\int_t^{t(p)} |k(x,t')|^2 dt' \right)^{1/2} (t(p) - t)^{1/2} \lesssim (t(p) - t)^{1/2}$$

which implies $|g_{ij}(x, t(p)) - g_{ij}(x,t)| < \frac{1}{2}\epsilon$ as long as δ_* is appropriately chosen.

[5]We use $\Phi_1 \lesssim \Phi_2$ to mean that $\Phi_1 \le C\Phi_2$ for some universal constant C.

The major part of the present paper is therefore to establish (1.27) under the weaker condition (1.11). To this end, we will use the Kirchoff parametrix to represent k as

$$-4\pi n(p)k(p)\cdot J=\int_{\mathcal{N}^-(p,\tau)}\Box k\cdot \mathbf{A}+\text{other terms},$$

for any $\delta<i_*(p,t)$, where J is any 2-covariant tensor at p tangent to $\Sigma_{t(p)}$ and $\mathbf{A}$ is the Σ-tangent tensor defined by

$$\mathbf{D}_L\mathbf{A}_{ij}+\frac{1}{2}\text{tr}\chi\mathbf{A}_{ij}=0\quad\text{on } \mathcal{N}^-(p,\tau),\qquad \lim_{t\to t(p)}(t(p)-t)\mathbf{A}_{ij}=J.$$

It can be shown that $\|r\mathbf{A}\|_{L^\infty(\mathcal{N}^-(p,\tau))}\lesssim 1$ together with other estimates on $\mathbf{A}$, where $r=\sqrt{(4\pi)^{-1}|S_t|}$ and $|S_t|$ denotes the area of S_t. Thus

$$n(p)|k(p)|\lesssim\int_{\mathcal{N}^-(p,\tau)}r^{-1}|\Box k|+\text{other terms}.$$

Next we let p move along an integral curve $\Phi(t)$ of $\mathbf{T}$ to get the representations of k at all points on this curve. Then we can reduce the proof of (1.27) to showing that

$$\int_{t(p)-\tau}^{t(p)}\left|\int_{\mathcal{N}^-(\Gamma(t),t-t(p)+\tau)}r^{-1}|\Box k|+\cdots\right|^2 dt\lesssim 1.$$

In view of (1.22), we have to employ various estimates of k and n on the null cones, which will be established by delicate analysis.

This paper is organized as follows. In Section 2, we collect some preliminary results related to the CMC foliation, which will be used frequently in the later sections. In Section 3, we establish various elliptic estimates on the lapse function n, in particular, we show that n can be bounded from below and above by positive universal constants. In Section 4, we provide the sketch of the proof of Theorem 1.2. We will explain how to use the bootstrap argument to establish (1.16) and other related estimates on the null cones. We then show how to use the estimate (1.27) to obtain a system of good local space-time coordinates which is crucial for completing the proof of Theorem 1.2. In order to establish (1.27), we derive a tensorial wave equation for k in Section 5 and provide the estimate for the so called k-flux in Section 6 which will be defined later. In Section 7 we provide some trace estimates on the surfaces S_t. We then use these results in Section 8 to establish various estimates for k, n and χ on the null cones. In section 9 we adapt the Kirchoff-Sobolev formula in [10] to represent the second fundamental form k along the null cones, through which we give the proof of (1.16) under the condition (1.11) and thus complete the proof of Theorem 1.2. Finally in Section 9 we complete the proof of Theorem 1.1.

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2. **Preliminaries**

For the lapse function n, by using the elliptic equation $-\Delta n + |k|^2 n = 1$, it follows easily from the maximum principle that

$$\frac{1}{\|k\|_{L^\infty(\Sigma_t)}} \le n \le \frac{3}{t^2} \quad \text{on } \Sigma_t. \tag{2.1}$$

Thus, if we knew that $\|k\|_{L^\infty(\Sigma_t)}$ is uniformly bounded with respect to $t \in [t_0, t_*)$, then we could get a positive uniform lower bound on n. Unfortunately, we only have the weaker assumption (**A1**) on k, which does not allow (2.1) to give a positive uniform lower bound on n directly. In the next section, we will show under the assumption (**A1**) that $C^{-1} \le n \le C$ on $\mathcal{M}_I$ for some universal constant $C > 0$.

For each slice Σ_t, we use $|\Sigma_t|$ to denote its volume. Then, by using $\partial_t g_{ij} = -2nk_{ij}$ and $\mathrm{Tr} k = t$ on Σ_t we have

$$\frac{d}{dt}\left(|t|^3 |\Sigma_t|\right) = \int_{\Sigma_t} t^2 \left(t^2 n - 3\right) d\mu_g \le 0.$$

This implies that $|t|^3|\Sigma_t|$ is decreasing with respect to t. Consequently

$$|\Sigma_t| \le \frac{|t_0|^3}{|t|^3}|\Sigma_{t_0}| \le \frac{|t_0|^3}{|t_*|^3}|\Sigma_{t_0}|, \quad \forall t_0 \le t \le t_*. \tag{2.2}$$

2.1. **Bel-Robinson Energy.** We start with a brief review of Bel-Robinson energy, one may consult [5] for more details. Associated to the Weyl tensor $\mathbf{R}$, the Bel-Robinson tensor is the full symmetric, traceless tensor defined by

$$\mathbf{Q}[\mathbf{R}]_{\alpha\beta\gamma\delta} = \mathbf{R}_{\alpha\lambda\gamma\mu}\mathbf{R}_{\beta}{}^{\lambda}{}_{\delta}{}^{\mu} + {}^\star\mathbf{R}_{\alpha\lambda\gamma\mu}{}^\star\mathbf{R}_{\beta}{}^{\lambda}{}_{\delta}{}^{\mu}. \tag{2.3}$$

Then $\mathbf{Q}[\mathbf{R}](X,Y,X,Y) \ge 0$ whenever X, Y are timelike vectors, with equality only if $\mathbf{R} = 0$. Let $\mathbf{P}_\alpha = \mathbf{Q}[\mathbf{R}]_{\alpha\beta\gamma\delta}\mathbf{T}^\beta\mathbf{T}^\gamma\mathbf{T}^\delta$. Since $\mathbf{R}_{\alpha\beta} = 0$, a straightforward calculation shows that

$$\mathbf{D}^\alpha \mathbf{P}_\alpha = -3\pi^{\alpha\beta}\mathbf{Q}[\mathbf{R}]_{\alpha\beta\gamma\delta}\mathbf{T}^\gamma\mathbf{T}^\delta. \tag{2.4}$$

If we introduce the Bel-Robinson energy $\mathcal{Q}(t)$ by

$$\mathcal{Q}(t) := \int_{\Sigma_t} \mathbf{Q}[\mathbf{R}](\mathbf{T},\mathbf{T},\mathbf{T},\mathbf{T}) d\mu_{\Sigma_t},$$

then, by integrating (2.4) in a slab $\mathcal{M}_J = \cup_{t\in J}\Sigma_t$ with $J = [t_0, t] \subset [t_0, t_*)$, we obtain

$$\mathcal{Q}(t) = \mathcal{Q}(t_0) - 3\int_{t_0}^t \int_{\Sigma_{t'}} n\mathbf{Q}[\mathbf{R}]_{\alpha\beta 00}\pi^{\alpha\beta} d\mu_{\Sigma_{t'}} dt'.$$

Let E and H denote the electric and magnetic parts of the curvature tensor $\mathbf{R}$ defined by

$$E(X,Y) = \mathbf{g}(\mathbf{R}(X,\mathbf{T})\mathbf{T},Y), \qquad H(X,Y) = \mathbf{g}({}^\star\mathbf{R}(X,\mathbf{T})\mathbf{T},Y) \tag{2.5}$$

with ${}^\star\mathbf{R}$ the Hodge dual of $\mathbf{R}$. It is well known that E and H are traceless symmetric 2-tensors tangent to Σ_t with

$$|\mathbf{R}|^2 = |E|^2 + |H|^2,$$
$$|\mathbf{Q}| \le 4(|E|^2 + |H|^2)$$

and

$$\mathbf{Q}(\mathbf{T},\mathbf{T},\mathbf{T},\mathbf{T}) = |E|^2 + |H|^2.$$

Therefore

$$\mathcal{Q}(t) \le \mathcal{Q}(t_0) + 12 \int_{t_0}^{t} \|n\pi\|_{L^\infty(\Sigma_{t'})} \mathcal{Q}(t') dt'.$$

By the Gronwall inequality it follows that

$$\mathcal{Q}(t) \le \mathcal{Q}(t_0) \exp\left(12 \int_{t_0}^{t} \|n\pi\|_{L^\infty(\Sigma_{t'})} dt'\right)$$

for all $t \in [t_0, t_*)$. Therefore, in view of the condition (**A1**) we obtain the uniform boundedness of the Bel-Robinson energy.

Lemma 2.1. *Under the condition* (**A1**)*, there exists a constant C depending only on $\mathcal{K}_0$ and t_* such that*

$$\mathcal{Q}(t) \le C Q_0^2$$

for all $t \in [t_0, t_*)$, where $Q_0^2 := \mathcal{Q}(t_0)$.

Consequently we have

Lemma 2.2. *Let the condition* (**A1**) *hold. Then on any CMC leaf $\Sigma_t \subset \mathcal{M}_*$ there holds*

$$\int_{\Sigma_t} \left(|\nabla k|^2 + \frac{1}{4}|k|^4\right) + \int_{\Sigma_t} |Ric|^2 \lesssim Q_0^2. \tag{2.6}$$

Proof. The inequality on k follows from [12, Proposition 8.4] and Lemma 2.1. The inequality on Ric then follows from the identity $R_{ij} - k_{ia}k^{aj} + \mathrm{Tr}k\, k_{ij} = E_{ij}$. □

2.2. **Harmonic coordinates.** For any coordinate chart $\mathcal{O} \subset \Sigma_0$ with local coordinates $x = (x^1, x^2, x^3)$, we denote by $x^0 = t, x^1, x^2, x^3$ the transported coordinates on $I \times \mathcal{O}$ obtained by transporting along the integral curves of $\mathbf{T}$. The following is an immediate consequence of (**A1**) and (1.2).

Proposition 1. *Let the assumption* (**A1**) *hold. There exists a positive constant C_0 depending only on $\mathcal{K}_0$ such that, relative to the induced transported coordinates $x^0 = t$, x^1, x^2, x^3 in $I \times \mathcal{O}$ we have*

$$C_0^{-1}|\xi|^2 \le g_{ij}(t,x)\xi^i\xi^j \le C_0|\xi|^2. \tag{2.7}$$

Proof. This is [12, Proposition 2.4] which was stated under the stronger condition (1.9), the proof however requires only the weaker assumption (**A1**). □

This proposition enables us to derive a uniform lower bound on the volume radius for all the slices Σ_t. Here, for a 3-dimensional Riemannian manifold (M, g), the volume radius $r_{vol}(p, \rho)$ at a point $p \in M$ and scales $\le \rho$ is defined by

$$r_{vol}(p, \rho) = \inf_{r \le \rho} \frac{|B_r(p)|}{r^3}$$

with $|B_r(p)|$ the volume of $B_r(p)$ relative to metric g. The volume radius $r_{vol}(M, \rho)$ of M on scales $\le \rho$ is the infimum of $r_{vol}(p, \rho)$ over all $p \in M$. Using Proposition 1, it has been show in [11, Proposition 4.4] that the volume radius $r_{vol}(\Sigma_t, 1)$ of each Σ_t on scales ≤ 1 verifies

$$r_{vol}(\Sigma_t, 1) \ge v_0$$

for some constant $v_0 > 0$ depending only on $\mathcal{K}_0$.

From the previous subsection we have already obtained, under (**A1**), that

$$\|Ric\|_{L^2(\Sigma_t)} \le C \quad \text{and} \quad |\Sigma_t| \le \frac{|t_0|^3}{|t_*|^3}|\Sigma_{t_0}|.$$

Therefore, Theorem 3.5 in [1] applies and provides the following results on the existence of harmonic coordinates.

Proposition 2. *Let the assumption* (**A1**) *hold. For any $\epsilon > 0$, there exists $r_0 > 0$ depending on ϵ, Q_0, $\mathcal{K}_0$, $|\Sigma_0|$ and t_* such that every geodesic ball $B_r(p) \subset \Sigma_t$ with $r \le r_0$ admits a system of harmonic coordinates $x = (x^1, x^2, x^3)$ under which*

$$(1+\epsilon)^{-1}\delta_{ij} \le g_{ij} \le (1+\epsilon)\delta_{ij} \tag{2.8}$$

$$r\int_{B_r(p)} |\partial^2 g_{ij}|^2 d\mu_g \le \epsilon. \tag{2.9}$$

We will not use the full strength of this result. The crucial part in our applications is the existence of a local coordinates $x = (x^1, x^2, x^3)$ on each $B_{r_0}(p) \subset \Sigma_t$ satisfying (2.8) with $r_0 > 0$ depending only on ϵ, Q_0, $\mathcal{K}_0$, $|\Sigma_0|$ and t_*.

2.3. **Sobolev-type inequalities.** We will give several Sobolev type inequalities under the assumption (**A1**). These inequalities are useful in establishing various estimates.

Lemma 2.3. *Let the assumption* (**A1**) *hold on $\mathcal{M}_*$. Then for any smooth tensor field F on $\Sigma_t \subset \mathcal{M}_*$ and any $2 \le p \le 6$ there holds*

$$\|F\|_{L^p(\Sigma_t)} \le C\left(\|\nabla F\|_{L^2(\Sigma_t)}^{3/2-3/p}\|F\|_{L^2(\Sigma_t)}^{3/p-1/2} + \|F\|_{L^2(\Sigma_t)}\right), \tag{2.10}$$

where C is a constant depending only on $\mathcal{K}_0$ and p.

Proof. This is [12, Corollary 2.7]. □

The following calculus inequality is useful in deriving L^∞ bounds of certain quantities.

Lemma 2.4. *Let the assumption* (**A1**) *hold on $\mathcal{M}_*$. Then for any smooth tensor field F on $\Sigma_t \subset \mathcal{M}_*$ and $3 < p \le 6$ there holds*

$$\|F\|_{L^\infty(\Sigma_t)} \le C\left(\|\nabla^2 F\|_{L^2(\Sigma_t)}^{3/2-3/p}\|\nabla F\|_{L^2(\Sigma_t)}^{3/p-1/2} + \|\nabla F\|_{L^2(\Sigma_t)} + \|F\|_{L^2(\Sigma_t)}\right),$$

where C is a constant depending only on $\mathcal{K}_0$ and p.

Proof. By using a partition of unity, the Sobolev embedding $W^{1,p}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$ with $p > 3$, and (2.7) in Proposition 1, it is easy to derive for any scalar function f on Σ_t that

$$\|f\|_{L^\infty(\Sigma_t)} \le C\left(\|\nabla f\|_{L^p(\Sigma_t)} + \|f\|_{L^p(\Sigma_t)}\right).$$

Now we take $f = |F|^2$ in the above inequality. It yields

$$\begin{aligned}\|F\|^2_{L^\infty(\Sigma_t)} &\le C\left(\|\nabla |F|^2\|_{L^p(\Sigma_t)} + \||F|^2\|_{L^p(\Sigma_t)}\right)\\ &\le C\left(\|\nabla F\|_{L^p(\Sigma_t)} + \|F\|_{L^p(\Sigma_t)}\right)\|F\|_{L^\infty(\Sigma_t)}.\end{aligned}$$

This implies for $p > 3$ that

$$\|F\|_{L^\infty(\Sigma_t)} \le C\left(\|\nabla F\|_{L^p(\Sigma_t)} + \|F\|_{L^p(\Sigma_t)}\right).$$

The desired inequality follows by applying Lemma 2.3 to the term $\|\nabla F\|_{L^p(\Sigma_t)}$. □

3. Elliptic estimates for the lapse function n

In this section, we establish a series of elliptic estimates on the lapse function n together with n^{-1} and $\dot{n} := \partial_t n$ under the assumption (**A1**). These results will be repeatedly used in later sections. Throughout this paper we will use C to denote a universal constant.

3.1. Estimates on n.

Proposition 3. *Let the assumption* (**A1**) *hold. Then on every* $\Sigma_t \subset \mathcal{M}_*$ *there holds*

$$\|\nabla^2 n\|_{L^2(\Sigma_t)} + \|\nabla n\|_{L^2(\Sigma_t)} \le C.$$

Proof. We multiply the equation $-\Delta n + |k|^2 n = 1$ by n and integrate over Σ_t to obtain

$$\int_{\Sigma_t} \left(|\nabla n|^2 + |k|^2 n^2\right) = \int_{\Sigma_t} n.$$

Since $0 < n \le 3/t^2 \le 3/t_*^2$ and $|\Sigma_t| \le |\Sigma_{t_0}||t_0|^3/|t_*|^3$, this immediately gives the desired bound on $\|\nabla n\|_{L^2(\Sigma_t)}$.

In order to obtain the bound on $\|\nabla^2 n\|_{L^2(\Sigma_t)}$, we use the Böchner identity

$$\int_{\Sigma_t} |\nabla^2 n|^2 = \int_{\Sigma_t} \left(|\Delta n|^2 - R_{ij}\nabla^i n \nabla^j n\right),$$

the equation $\Delta n = |k|^2 n - 1$, Lemma 2.2 and the Hölder inequality to infer that

$$\|\nabla^2 n\|_{L^2} \lesssim \|k\|_{L^4}^2 + |\Sigma_t|^{\frac{1}{2}} + \|Ric\|_{L^2}^{\frac{1}{2}} \|\nabla n\|_{L^4} \lesssim 1 + \|\nabla n\|_{L^4}.$$

With the help of Lemma 2.2, we have

$$\|\nabla^2 n\|_{L^2} \lesssim 1 + \|\nabla^2 n\|_{L^2}^{3/4} \|\nabla n\|_{L^2}^{1/4} + \|\nabla n\|_{L^2}.$$

Therefore

$$\|\nabla^2 n\|_{L^2} \lesssim 1 + \|\nabla n\|_{L^2} \lesssim 1$$

and the proof is complete. □

Proposition 4. *Let the assumption* (**A1**) *hold. Then there hold*

$$\|\nabla^3 n\|_{L_t^1 L_x^2(\mathcal{M}_*)} \le C \tag{3.1}$$

$$\|\nabla n\|_{L_t^b L_x^\infty(\mathcal{M}_*)} \le C \tag{3.2}$$

where $1 \le b < 2$.

We will give the proof with the help of the following lemma.

Lemma 3.1. *Let the assumption* (**A1**) *hold. Then for any* 1*-form* F *on* $\Sigma_t \subset \mathcal{M}_*$ *we have*

$$\|\nabla^2 F\|_{L^2(\Sigma_t)} \le C \left(\|\Delta F\|_{L^2(\Sigma_t)} + \|\nabla F\|_{L^2(\Sigma_t)} + \|F\|_{L^2(\Sigma_t)}\right). \tag{3.3}$$

Proof. It is well known that for any 1-form F on Σ_t there holds the Böchner identity

$$\begin{aligned}\int_{\Sigma_t} |\Delta F|^2 = \int_{\Sigma_t} |\nabla^2 F|^2 - \frac{1}{2}\int_{\Sigma_t} R_{diac} R_{miac} F_d F_m \\ + \int_{\Sigma_t} R_{ad} \nabla_d F_i \nabla_a F_i - \int_{\Sigma_t} R_{idac} \nabla_c F_d \nabla_a F_i.\end{aligned} \tag{3.4}$$

Since Σ_t is 3-dimensional, the Riemannian curvature tensor is completely determined by its Ricci curvature, i.e.

$$R_{idac} = g_{ia}R_{dc} + g_{dc}R_{ia} - R_{ic}g_{da} - R_{da}g_{ic} - \frac{1}{2}(g_{ia}g_{dc} - g_{ic}g_{da})R.$$

Thus, we may use (3.4), the Hölder inequality, Lemma 2.2, Lemma 2.3 and Lemma 2.4 to obtain the estimate

$$\begin{aligned}\|\nabla^2 F\|_{L^2} &\lesssim \|\Delta F\|_{L^2} + \|Ric\|_{L^2}^{1/2}\|\nabla F\|_{L^4} + \|F\|_{L^\infty}\|Ric\|_{L^2} \\ &\lesssim \|\Delta F\|_{L^2} + \left(\|\nabla^2 F\|_{L^2}^{3/4}\|\nabla F\|_{L^2}^{1/4} + \|\nabla F\|_{L^2}\right).\end{aligned}$$

With the help of Young's inequality, the inequality (3.3) follows immediately. □

Proposition 5. *Let the assumption* (**A1**) *hold. Then on every* $\Sigma_t \subset \mathcal{M}_*$ *there hold*

$$\|\nabla^3 n\|_{L^2(\Sigma_t)} \le C\left(\|\nabla n\|_{H^1(\Sigma_t)} + \|k\|_{L^\infty(\Sigma_t)}\right), \tag{3.5}$$

$$\|\nabla n\|_{L^\infty(\Sigma_t)} \le C\left(\|\nabla n\|_{H^1(\Sigma_t)} + \|k\|_{L^\infty(\Sigma_t)}^{3/2-3/p}\|\nabla^2 n\|_{L^2(\Sigma_t)}^{3/p-1/2}\right), \tag{3.6}$$

where $3 < p \le 6$.

Proof. A simple application of Lemma 3.1 to $F = \nabla n$ gives

$$\|\nabla^3 n\|_{L^2} \lesssim \|\Delta\nabla n\|_{L^2} + \|\nabla^2 n\|_{L^2} + \|\nabla n\|_{L^2}. \tag{3.7}$$

Recall the commutation formula $\Delta\nabla_i n = \nabla_i\Delta n + R_{ij}\nabla_j n$ and the equation $-\Delta n + |k|^2 n = 1$, we can estimate

$$\|\Delta\nabla n\|_{L^2} \lesssim \|k\|_{L^6}^2\|\nabla n\|_{L^6} + \|k\|_{L^\infty}\|\nabla k\|_{L^2} + \|Ric\|_{L^2}\|\nabla n\|_{L^\infty}.$$

Plugging this into (3.7), using Lemma 2.2 and Lemma 2.3 gives

$$\|\nabla^3 n\|_{L^2} \lesssim \|\nabla n\|_{L^\infty} + \|\nabla n\|_{H^1} + \|k\|_{L^\infty}.$$

Using Lemma 2.4 for the term $\|\nabla n\|_{L^\infty}$ with $p = 4$, we then obtain

$$\|\nabla^3 n\|_{L^2} \lesssim \|\nabla^3 n\|_{L^2}^{3/4}\|\nabla^2 n\|_{L^2}^{1/4} + \|\nabla n\|_{H^1} + \|k\|_{L^\infty}.$$

This clearly implies (3.5). The inequality (3.6) is an immediate consequence of (3.5) and Lemma 2.4. □

Proposition 4 follows by integrating (3.5) and (3.6) in time with the help of (**A1**) and Proposition 3.

3.2. Estimates on n^{-1}.

Proposition 6. *Let the assumption* (**A1**) *hold. Then on each* $\Sigma_t \subset \mathcal{M}_*$ *there hold*

$$\|\nabla^2(n^{-1})\|_{L^2(\Sigma_t)} + \|n^{-1}\|_{L^\infty(\Sigma_t)} \le C.$$

Proof. We first have from the Bochner identity that

$$\begin{aligned}\int_\Sigma |\nabla^2(n^{-1})|^2 &= \int_\Sigma |\Delta(n^{-1})|^2 - \int_\Sigma R_{ij}\nabla_i(n^{-1})\nabla_j(n^{-1}) \\ &\le \|\Delta(n^{-1})\|_{L^2}^2 + \|Ric\|_{L^2}\|\nabla(n^{-1})\|_{L^4}^2.\end{aligned} \tag{3.8}$$

Since $-\Delta n + |k|^2 n = 1$, we have

$$\Delta(n^{-1}) = 2n^{-3}|\nabla n|^2 + n^{-2} - |k|^2 n^{-1}. \tag{3.9}$$

Consequently, it follows from the Hölder inequality that

$$\|\Delta(n^{-1})\|_{L^2} \lesssim \|n^{-1}\nabla n\|_{L^4}\|\nabla(n^{-1})\|_{L^4} + \|k\|_{L^6}^2\|n^{-1}\|_{L^6} + \|n^{-1}\|_{L^4}^2.$$

Combining this inequality with (3.8) and using the Sobolev embedding $H^1(\Sigma) \hookrightarrow L^p(\Sigma)$ with $2 \le p \le 6$, which is a consequence of Lemma 2.3, we obtain

$$\begin{aligned}\|\nabla^2(n^{-1})\|_{L^2} &\lesssim \|n^{-1}\nabla n\|_{L^4}\|\nabla(n^{-1})\|_{L^4} + \left(\|n^{-1}\|_{H^1} + \|k\|_{L^6}^2\right)\|n^{-1}\|_{H^1}\\ &\quad + \|Ric\|_{L^2}^{\frac{1}{2}}\|\nabla(n^{-1})\|_{L^4}\end{aligned} \tag{3.10}$$

We need to estimate $\|n^{-1}\nabla n\|_{L^4}$. To this end, we multiply the equation $-\Delta n + |k|^2 n = 1$ by n^{-l} for some positive integer l and then integrate by parts over Σ_t to obtain

$$\int_{\Sigma_t}\left(l n^{-l-1}|\nabla n|^2 + n^{-l}\right) = \int_{\Sigma_t} n^{-l+1}|k|^2. \tag{3.11}$$

Taking $l = 7$ gives

$$\int_{\Sigma_t} n^{-8}|\nabla n|^2 \lesssim \int_{\Sigma_t} n^{-6}|k|^2 \lesssim \|k\|_{L^4}^2\|n^{-2}\|_{L^6}^3.$$

Therefore

$$\|n^{-1}\nabla n\|_{L^4} \le \left(\int_{\Sigma_t} n^{-8}|\nabla n|^2\right)^{1/8}\left(\int_{\Sigma_t}|\nabla n|^6\right)^{1/8} \lesssim \|k\|_{L^4}^{1/4}\|n^{-2}\|_{L^6}^{3/8}\|\nabla n\|_{L^6}^{3/4}.$$

By Lemma 2.3 and Proposition 3, we have $\|\nabla n\|_{L^6} \le C\left(\|\nabla^2 n\|_{L^2} + \|\nabla n\|_{L^2}\right) \le C$. By using Lemma 2.3 and (3.11) with $l = 5$ we also have

$$\begin{aligned}\|n^{-2}\|_{L^6} \lesssim \|n^{-2}\|_{H^1} &\lesssim \left(\int_{\Sigma_t} n^{-4}|k|^2\right)^{1/2} + \|n^{-1}\|_{L^4}^2\\ &\lesssim \|k\|_{L^6}\|n^{-1}\|_{L^6}^2 + \|n^{-1}\|_{L^4}^2\\ &\lesssim \left(1 + \|k\|_{L^6}\right)\|n^{-1}\|_{H^1}^2.\end{aligned}$$

Therefore

$$\|n^{-1}\nabla n\|_{L^4} \lesssim \left(1 + \|k\|_{L^6}^{3/8}\right)\|k\|_{L^4}^{1/4}\|n^{-1}\|_{H^1}^{3/4}.$$

Combining this inequality with (3.10) and using Lemma 2.2 to bound $\|k\|_{L^4}$, $\|k\|_{L^6}$ and $\|Ric\|_{L^2}$, it yields

$$\|\nabla^2(n^{-1})\|_{L^2} \lesssim \|n^{-1}\|_{H^1}^{3/4}\|\nabla(n^{-1})\|_{L^4} + (\|n^{-1}\|_{H^1} + 1)\|n^{-1}\|_{H^1} + \|\nabla(n^{-1})\|_{L^4}.$$

Applying Lemma 2.3 to the term $\|\nabla(n^{-1})\|_{L^4}$ gives

$$\begin{aligned}\|\nabla^2(n^{-1})\|_{L^2} &\lesssim \|n^{-1}\|_{H^1}^{3/4}\left(\|\nabla^2(n^{-1})\|_{L^2}^{3/4}\|\nabla(n^{-1})\|_{L^2}^{1/4} + \|\nabla(n^{-1})\|_{L^2}\right) + \|n^{-1}\|_{H^1}^2\\ &\quad + \|\nabla^2(n^{-1})\|_{L^2}^{3/4}\|\nabla(n^{-1})\|_{L^2}^{1/4} + \|n^{-1}\|_{H^1}.\end{aligned}$$

With the help of Young's inequality, we obtain

$$\|\nabla^2(n^{-1})\|_{L^2} \lesssim \|n^{-1}\|_{H^1}^4 + \|n^{-1}\|_{H^1}. \tag{3.12}$$

In order to estimate $\|n^{-1}\|_{H^1}$, we use (3.11) with $l = 3$ to obtain

$$\int_{\Sigma_t}\left(3|\nabla(n^{-1})|^2 + n^{-3}\right) = \int_{\Sigma_t}|k|^2 n^{-2}.$$

It then follows from the Hölder inequality and Lemma 2.3 that

$$\|\nabla(n^{-1})\|_{L^2} \lesssim \|k\|_{L^4}\|n^{-1}\|_{L^4} \lesssim \|k\|_{L^4}\left(\|\nabla(n^{-1})\|_{L^2}^{3/4}\|n^{-1}\|_{L^2}^{1/4} + \|n^{-1}\|_{L^2}\right).$$

This clearly implies

$$\|\nabla(n^{-1})\|_{L^2} \lesssim \left(\|k\|_{L^4} + \|k\|_{L^4}^4\right)\|n^{-1}\|_{L^2} \lesssim \|n^{-1}\|_{L^2}. \tag{3.13}$$

The combination of (3.12) and (3.13) gives

$$\|\nabla^2(n^{-1})\|_{L^2} + \|\nabla(n^{-1})\|_{L^2} \lesssim \|n^{-1}\|_{L^2}^4 + \|n^{-1}\|_{L^2}.$$

Note that (3.11) with $l = 2$ gives

$$\|n^{-1}\|_{L^2}^2 \le \int_{\Sigma_t} n^{-1}|k|^2 \le \|k\|_{L^4}^2\|n^{-1}\|_{L^2}.$$

This implies $\|n^{-1}\|_{L^2} \le \|k\|_{L^4}^2 \le C$. We therefore obtain $\|n^{-1}\|_{H^2} \le C$. With the help of Lemma 2.4 the estimate $\|n^{-1}\|_{L^\infty} \le C$ follows immediately. □

3.3. **Derivative estimates about $\dot{n}$.** In this subsection we will give various estimates on the derivative $\dot{n} := \partial_t n$. We start with deriving an elliptic equation for $\dot{n}$. By straightforward calculation we have

$$\Delta\dot{n} = -\dot{g}^{ij}\nabla_i\nabla_j n + \partial_t(\Delta n) + g^{ij}\dot{\Gamma}_{ij}^a\nabla_a n \tag{3.14}$$

Recall that

$$\dot{\Gamma}_{ij}^a = \frac{1}{2}g^{ab}(\nabla_i\dot{g}_{jb} + \nabla_j\dot{g}_{ib} - \nabla_b\dot{g}_{ij}).$$

From (1.2), (1.6) and the fact $\mathrm{Tr}k = t$ it then follows

$$g^{ij}\dot{\Gamma}_{ij}^a\nabla_a n = -2k_i^a\nabla^i n\nabla_a n + \mathrm{Tr}k|\nabla n|^2.$$

Plugging this identity into (3.14) and using $\dot{g}^{ij} = 2nk^{ij}$ and $\Delta n = |k|^2 n - 1$ we obtain

$$\Delta\dot{n} = -2nk^{ij}\nabla_i\nabla_j n + |k|^2\dot{n} + \partial_t(|k|^2)n - 2k_i^a\nabla^i n\nabla_a n + \mathrm{Tr}k|\nabla n|^2$$

We may use the equations (1.2) and (1.3) to derive

$$\partial_t(|k|^2) = -2k^{ij}\nabla_i\nabla_j n + 2nR_{ij}k^{ij} + 2n|k|^2\mathrm{Tr}k.$$

Consequently, we obtain

$$\begin{aligned}\Delta\dot{n} = &-4nk^{ij}\nabla_i\nabla_j n + |k|^2\dot{n} - 2k_i^a\nabla^i n\nabla_a n + \mathrm{Tr}k|\nabla n|^2\\ &+ 2nR_{ij}k^{ij} + 2n|k|^2\mathrm{Tr}k.\end{aligned} \tag{3.15}$$

Now we multiply the equation (3.15) by $\dot{n}$ and integrate over Σ_t, by using the boundedness of n and the Hölder inequality we obtain

$$\begin{aligned}&\int_{\Sigma_t}\left(|\nabla\dot{n}|^2 + |k|^2|\dot{n}|^2\right)\\ &\quad\lesssim \int_{\Sigma_t}\left(|\dot{n}||k||\nabla^2 n| + |\dot{n}||k||\nabla n|^2 + |\dot{n}||Ric||k| + |\dot{n}||k|^3\right)\\ &\quad\le \left(\|\nabla^2 n\|_{L^2} + \|\nabla n\|_{L^4}^2 + \|Ric\|_{L^2}\right)\|k\|_{L^4}\|\dot{n}\|_{L^4} + \|k\|_{L^6}^3\|\dot{n}\|_{L^2}.\end{aligned}$$

Using the bounds derived in Lemma 2.2 and Proposition 3 together with the Sobolev embedding we have

$$\int_{\Sigma_t}\left(|\nabla\dot{n}|^2 + |k|^2|\dot{n}|^2\right) \lesssim \|\dot{n}\|_{L^4} + \|\dot{n}\|_{L^2} \lesssim \|\nabla\dot{n}\|_{L^2} + \|\dot{n}\|_{L^2}.$$

Recall that $|k|^2 = |\hat{k}|^2 + t^2/3$ and $|t| \ge |t_*| > 0$. Therefore

$$\|\nabla \dot{n}\|_{L^2}^2 + \|\dot{n}\|_{L^2}^2 \lesssim \|\nabla \dot{n}\|_{L^2} + \|\dot{n}\|_{L^2}.$$

We therefore obtain

Lemma 3.2. *Let the assumption* (**A1**) *hold. Then for each* $\Sigma_t \subset \mathcal{M}_*$*, there holds*

$$\|\nabla \dot{n}\|_{L^2(\Sigma_t)} + \|\dot{n}\|_{L^2(\Sigma_t)} \le C. \tag{3.16}$$

Now we are ready to give some estimates on the mixed norms of $\dot{n}$.

Proposition 7. *Let the assumption* (**A1**) *hold. Let* $\dot{n} = \partial_t n$*. Then there hold*

$$\|\nabla^2 \dot{n}\|_{L_t^1 L_x^2(\mathcal{M}_*)} \le C \qquad \textit{and} \qquad \|\dot{n}\|_{L_t^b L_x^\infty(\mathcal{M}_*)} \le C$$

for any $1 \le b < 2$.

Proof. In view of the assumption (**A1**), it suffice to establish on every Σ_t the inequalities

$$\|\nabla^2 \dot{n}\|_{L^2(\Sigma_t)} \lesssim C \left(\|k\|_{L^\infty(\Sigma_t)} + 1\right), \tag{3.17}$$

and

$$\|\dot{n}\|_{L^\infty(\Sigma_t)} \le C \left(\|k\|_{L^\infty(\Sigma_t)}^{3/2-3/p} + 1\right) \tag{3.18}$$

for any $3 < p \le 6$.

By the Böchner identity, we have

$$\|\nabla^2 \dot{n}\|_{L^2}^2 \le \|\Delta \dot{n}\|_{L^2}^2 + \|Ric\|_{L^2} \|\nabla \dot{n}\|_{L^4}^2.$$

By using $\|Ric\|_{L_x^2} \lesssim 1$ and applying Lemma 2.3 to $\|\nabla \dot{n}\|_{L^4}$ we obtain

$$\|\nabla^2 \dot{n}\|_{L^2} \lesssim \|\Delta \dot{n}\|_{L^2} + \|\nabla \dot{n}\|_{L^2}^{3/4} \|\nabla \dot{n}\|_{L^2}^{1/4} + \|\nabla \dot{n}\|_{L^2}.$$

In view of Young's inequality and (3.16), it follows

$$\|\nabla^2 \dot{n}\|_{L^2} \lesssim \|\Delta \dot{n}\|_{L^2} + 1. \tag{3.19}$$

From the equation (3.15) it follows that

$$\|\Delta \dot{n}\|_{L^2} \lesssim \|k\|_{L^\infty}\| \left(\|\nabla^2 n\|_{L^2} + \|Ric\|_{L^2}\right) + \|k\|_{L^6}^2 \|\dot{n}\|_{L^6} + \|\nabla n\|_{L^6}^2 \|k\|_{L^6} + \|k\|_{L^6}^3.$$

With the help of the estimates derived in Lemma 2.2, Proposition 3 and (3.16) together with the Sobolev embedding we have $\|\Delta \dot{n}\|_{L^2} \lesssim \|k\|_{L^\infty} + 1$. Therefore $\|\nabla^2 \dot{n}\|_{L^2} \lesssim \|k\|_{L^\infty} + 1$ which is exactly (3.17). The inequality (3.18) immediately follows from Lemma 2.4, (3.17) and (3.16). □

4. Null radius of injectivity: proof of main theorem II

In this section we will give the sketch of the proof of Theorem 1.2. The complete proof is rather involved and requires a delicate bootstrap argument. For any $t_0 < t_1 < t_*$ we consider the slab $\mathcal{M}_I = \cup_{t \in I} \Sigma_t$ with $I = [t_0, t_1]$. We set, for each $p \in \mathcal{M}_I$,

$$\tilde{i}_*(p,t) = \begin{cases} +\infty, & \text{if } i_*(p,t) > t(p) - t_0, \\ i_*(p,t), & \text{otherwise} \end{cases}$$

and define

$$i_* := \min\{\tilde{i}_*(p,t) : p \in \mathcal{M}_I\}. \tag{4.1}$$

Due to the compactness of $\mathcal{M}_I$, we have $i_* > 0$. In order to complete the proof of Theorem 1.2, it suffices to show that $i_* > \delta_*$ for some universal constant $\delta_* > 0$.

We will use the following result concerning the lower bound on the null radius of injectivity of a globally hyperbolic space-time which has essentially been proved in [11].

Theorem 4.1. *Let $C^{-1} \le n \le C$ on $\mathcal{M}_I$ for some constant $C > 0$. Then there exists a small constant $\epsilon > 0$ depending only on C such that if, for some constant $\delta_* > 0$, the following three conditions hold for all $p \in \mathcal{M}_I$:*

C1. *the null radius of conjugacy satisfies*

$$s_*(p,t) > \min\{i_*, \delta_*\};$$

C2. *for each t satisfying*

$$0 \le t(p) - t \le \min\{i_*, \delta_*\},$$

the metric γ_t on $\mathbb{S}^2$, obtained by restricting the metric g on Σ_t to $S_t := \mathcal{N}^-(p) \cap \Sigma_t$ and then pulling it back to $\mathbb{S}^2$ by the exponential map $\mathcal{G}(t,\cdot)$, verifies

$$|\gamma_t(X,X) - \overset{\circ}{\gamma}(X,X)| < \epsilon \overset{\circ}{\gamma}(X,X), \quad \forall X \in T\mathbb{S}^2,$$

where $\overset{\circ}{\gamma}$ is the standard metric on $\mathbb{S}^2$;

C3. *On $\mathcal{U}_p := I_p \times B_{\delta_*}(p)$ with $I_p := [t(p) - \min\{i_*, \delta_*\}, t(p)]$ and $B_{\delta_*}(p) \subset \Sigma_{t(p)}$ a geodesic ball, there is a system of coordinates x^α with $x^0 = t$ relative to which the metric $\mathbf{g}$ is close to the Minkowski metric $\mathbf{m}_{\alpha\beta} = -n(p)dt^2 + \delta_{ij}dx^i dx^j$ in the sense that*

$$|n - n(p)| + |g_{ij} - \delta_{ij}| < \epsilon \quad \text{on } \mathcal{U}_p,$$

then there holds $i_* > \delta_*$, i.e. the null radius of injectivity verifies

$$i_*(p,t) > \min\{\delta_*, t(p) - t_0\}$$

for all $p \in \mathcal{M}_I$.

Let us briefly outline the idea of proof. Assume that $i_* \le \delta_*$. Let $p_0 \in \mathcal{M}_I$ be a point such that $i_*(p_0,t) = i_* \le t(p) - t_0$. By **C1** we have $s_*(p_0,t) > i_*(p_0,t) = l_*(p_0,t)$. Thus there exist two distinct past null geodesics γ_1 and γ_2 initiating at p_0 intersect at a point q_0 with $t(q_0) = t(p_0) - i_*$. According to the definition of i_* and [11, Lemma 3.1] γ_1 and γ_2 are opposite at both p_0 and q_0. On the other hand, under the conditions **C2** and **C3**, Lemma 3.2 and Lemma 3.3 in [11] imply that such two null geodesics can not intersect in the time slab $[t(p_0) - i_*, t(p_0)]$.

Theorem 4.1 provides a general framework to estimate the null radius of injectivity from below. Under the condition (1.9), in [11] Klainerman and Rodnianski showed that the conditions **C1**–**C3** hold with a universal constant $\delta_* > 0$; thus they derived a universal lower bound on the null radius of injectivity.

In the following we will describe how to verify the conditions **C1**–**C3** under the assumption (**A1**). To this end, for each $p \in \mathcal{M}_I$ consider the past null cone $\mathcal{N}^-(p)$, let s be its affine parameter and let $S_t = \mathcal{N}^-(p) \cap \Sigma_t$. Then S_t is diffeomorphic to $\mathbb{S}^2$ for each t satisfying $t(p) - i_*(p,t) < t < t(p)$. Let γ be the restriction of $\mathbf{g}$ to S_t and let $|S_t|$ be the corresponding area. The radius of S_t is defined to be

$$r := \sqrt{(4\pi)^{-1}|S_t|} \tag{4.2}$$

which is a function of t only.

On $\mathcal{N}^-(p,\tau)\setminus\{p\}$ with $\tau < i_*(p,t)$ we can define a conjugate null vector $\underline{L}$ with $\mathbf{g}(L,\underline{L}) = -2$ and such that $\underline{L}$ is orthogonal to the leafs S_t. In addition we can choose $(e_A)_{A=1,2}$ tangent to S_t such that $(e_A)_{A=1,2}$, $e_3 = \underline{L}$, $e_4 = L$ form a null frame, i.e.

$$\mathbf{g}(L,\underline{L}) = -2, \quad \mathbf{g}(L,L) = \mathbf{g}(\underline{L},\underline{L}) = \mathbf{g}(L,e_A) = \mathbf{g}(\underline{L},e_A) = 0, \quad \mathbf{g}(e_A,e_B) = \delta_{AB}.$$

The null second fundamental forms χ, $\underline{\chi}$, the torsion ζ and the Ricci coefficient $\underline{\zeta}$ of the foliation S_t are then defined as follows

$$\chi_{AB} = \mathbf{g}(\mathbf{D}_A L, e_B), \qquad \underline{\chi}_{AB} = \mathbf{g}(\mathbf{D}_A \underline{L}, e_B),$$

$$\zeta_A = \frac{1}{2}\mathbf{g}(\mathbf{D}_A L, \underline{L}), \qquad \underline{\zeta}_A = \frac{1}{2}\mathbf{g}(e_A, \mathbf{D}_L \underline{L}).$$

In addition we define

$$\mathrm{tr}\chi = \gamma^{AB}\chi_{AB}, \qquad \hat{\chi}_{AB} = \chi_{AB} - \frac{1}{2}\mathrm{tr}\chi\,\gamma_{AB}.$$

We can define $\mathrm{tr}\underline{\chi}$ and $\hat{\underline{\chi}}$ similarly.

We introduce the null lapse function

$$a^{-1} := \mathbf{g}(L,\mathbf{T}).$$

Then $a > 0$ and $a(p) = 1$. It is easy to see that

$$L = -a^{-1}(\mathbf{T}+N), \qquad \underline{L} = -a(\mathbf{T}-N),$$

where N denotes the unit inward normal to S_t in Σ_t. We also introduce the function

$$\nu := -n^{-1}\nabla_N n + k_{NN}$$

which is relevant to the estimate on a.

For any S_t-tangent tensor field F we define the norm $\|F\|_{L^\infty_\omega L^2_t(\mathcal{N}^-(p,\tau))}$ by

$$\|F\|^2_{L^\infty_\omega L^2_t(\mathcal{N}^-(p,\tau))} := \sup_{\omega\in\mathbb{S}^2}\int_{t(p)-\tau}^{t(p)} |F|^2 nadt := \sup_{\omega\in\mathbb{S}^2}\int_{\Gamma_\omega}|F|^2 nadt,$$

where Γ_ω denotes the portion of a past null geodesic initiating from p contained in $\mathcal{N}^-(p,\tau)$.

The following result is sufficient to prove the conditions **C1**–**C3** in Theorem 4.1.

Theorem 4.2. *Let the assumption* (**A1**) *hold. Then there exist universal constants* $\delta_* > 0$ *and* $C_* > 0$ *such that for any* $p \in \mathcal{M}_I$ *there hold*

$$\int_{t(p)-\tau}^{t(p)} |k(\Phi(t))|^2 dt \le C_* \tag{4.3}$$

with Φ *the integral curve of* $\mathbf{T}$ *through* p, *and*

$$|a-1| \le \frac{1}{2}, \quad \left|tr\chi - \frac{2}{s}\right| \le C_*, \quad \|\hat{\chi}\|^2_{L^\infty_\omega L^2_t(\mathcal{N}^-(p,\tau))} \le C_* \tag{4.4}$$

on any null cones $\mathcal{N}^-(p,\tau)$, *where* $\tau := \min\{i_*,\delta_*\}$.

In fact, the estimate on $\mathrm{tr}\chi$ in (4.4) implies the condition **C1**, see [6, 4]. Next we will show that the estimates in (4.4) imply the condition **C2**. To see this, we recall that $\frac{ds}{dt} = -na$ and $\frac{d}{ds}\gamma_{AB} = 2\chi_{AB}$. Then

$$\frac{d}{dt}(s^{-2}\gamma_{AB}) = -na\left(-2s^{-3}\gamma_{AB} + 2s^{-2}\chi_{AB}\right)$$

Let $X \in T\mathbb{S}^2$ be any vector field. We integrate the above equation along any null geodesic and note that $\lim_{t\to t(p)^-} s(t)^{-2}\gamma(t) = \overset{\circ}{\gamma}$,(see [13]), it follows that

$$\left|s(t)^{-2}\gamma(X,X) - \overset{\circ}{\gamma}(X,X)\right| \le \int_t^{t(p)} \left(2|\hat{\chi}| + \left|\mathrm{tr}\chi - \frac{2}{s(t')}\right|\right) s(t')^{-2}\gamma(X,X) n a dt'$$

Let

$$\Theta := 2|\hat{\chi}| + \left|\mathrm{tr}\chi - \frac{2}{s}\right|.$$

We then have

$$\begin{aligned}\left|s(t)^{-2}\gamma(X,X) - \overset{\circ}{\gamma}(X,X)\right| \le& \int_t^{t(p)} \Theta \left|s(t')^{-2}\gamma(X,X) - \overset{\circ}{\gamma}(X,X)\right| n a dt' \\ &+ \overset{\circ}{\gamma}(X,X)\int_t^{t(p)} \Theta(t') n a dt'.\end{aligned}$$

Therefore, it follows from the Gronwall inequality that

$$\left|s(t)^{-2}\gamma(X,X) - \overset{\circ}{\gamma}(X,X)\right| \le \overset{\circ}{\gamma}(X,X)\int_t^{t(p)} \Theta n a dt' \exp\left(\int_t^{t(p)} \Theta(t') n a dt'\right).$$

Since $0 < n \le 3/t_*^2$, the estimate (4.4) in Theorem 4.2 implies

$$\int_t^{t(p)} \Theta n a dt' \le C\left((t(p)-t)^{1/2} + (t(p)-t)\right) \le C(t(p)-t)^{1/2}$$

and consequently

$$\left|s^{-2}\gamma(X,X) - \overset{\circ}{\gamma}(X,X)\right| \le C(t(p)-t)^{1/2}\, \overset{\circ}{\gamma}(X,X) \tag{4.5}$$

for all $t(p) - \min\{i_*, \delta_*\} \le t < t(p)$, where C is a universal constant. The condition **C2** is thus verified.

The verification of the condition **C3**, using the estimate (4.3), is given in the following result.

Lemma 4.3. *Let the assumption* (**A1**) *hold. For any $\epsilon > 0$, there exists a constant $\delta_* > 0$ depending only on Q_0, $\mathcal{K}_0$, t_* and ϵ such that for every point $p \in \mathcal{M}_I$ there exists on $\mathcal{U}_p := I_p \times B_{\delta_*}(p)$ with $I_p = [t(p) - \min\{i_*, \delta_*\}, t(p)]$ a system of transported coordinates $t, x = (x^1, x^2, x^3)$ relative to which $\mathbf{g}$ is close to the Minkowski metric $\mathbf{m}(p) = -n(p)^2 dt^2 + \delta_{ij} dx^i dx^j$, in the sense that*

$$|g_{ij} - \delta_{ij}| < \epsilon \qquad \textit{and} \qquad |n - n(p)| < \epsilon. \tag{4.6}$$

Proof. It follows from Proposition 2 that there exists a constant $\delta_0 > 0$ depending only $\mathcal{K}_0$, Q_0, t_* and ϵ such that every geodesic ball $B_{\delta_0}(p) \subset \Sigma_{t(p)}$ admits a system of harmonic coordinates $x = (x^1, x^2, x^3)$ under which

$$(1+\epsilon/2)^{-1}\delta_{ij} \le g_{ij} \le (1+\epsilon/2)\delta_{ij}. \tag{4.7}$$

Under the transported coordinates $t, x = (x^1, x^2, x^3)$, let $p = (t(p), 0)$ and let $q = (t, x)$ be an arbitrary point in $I_p \times B_{\delta_*}(p)$ with $I_p = [t(p) - \min\{i_*, \delta_*\}, t(p)]$, where $0 < \delta_* \le \delta_0$ is a constant to be determined. By using the equation $\partial_t g_{ij} = -2nk_{ij}$ we have

$$|g_{ij}(t,x) - g_{ij}(t(p),x)| = \left|\int_t^{t(p)} \partial_t g_{ij}(t',x) dt'\right| = 2\int_t^{t(p)} n|k| dt'.$$

Using the bound $0 < n \le 3/t_*$, the Hölder inequality and the estimate (4.3) in Theorem 4.2, it follows for some universal constant $C_1 > 0$ that

$$|g_{ij}(t,x) - g_{ij}(t(p),x)| \le C_1 (t(p)-t)^{1/2} \le C_1 \delta_*^{1/2}.$$

In view of (4.7), we thus obtain

$$|g_{ij}(t,x) - \delta_{ij}| \le |g_{ij}(t,x) - g_{ij}(t(p),x)| + |g_{ij}(t(p),x) - \delta_{ij}| \le C_1 \delta_*^{1/2} + \frac{\epsilon}{2}, \tag{4.8}$$

which gives the first inequality in (4.6) by letting $C_1 \delta_*^{1/2} < \epsilon/2$.

Next we prove the second inequality in (4.6). From Proposition 7 we have

$$|n(t,x) - n(t(p),x)| \le \int_t^{t(p)} |\dot{n}(t',x)| dt' \le (t(p)-t)^{1/4} \|\dot{n}\|_{L_t^{4/3} L_x^\infty} \le C_2 \delta_*^{1/4},$$

while by employing Morrey's estimate, Lemma 2.3 and Proposition 3 we have

$$\begin{aligned}
|n(t(p),x) - n(t(p),0)| &\le C_2 \delta_*^{1/4} \|\nabla n\|_{L^4(\Sigma_{t(p)})} \\
&\le C_2 \delta_*^{1/4} \left(\|\nabla^2 n\|_{L^2}^{3/4} \|\nabla n\|_{L^2}^{1/4} + \|\nabla n\|_{L_x^2} \right) \\
&\le C_2 \delta_*^{1/4},
\end{aligned}$$

where $C_2 > 0$ is a universal constant. Therefore

$$|n(t,x) - n(p)| \le 2C_2 \delta_*^{1/4}$$

which implies the second inequality in (4.3) by further letting $2C_2 \delta_*^{1/4} < \epsilon$. $\square$

The proof of Theorem 4.2 is based on a delicate bootstrap argument. We first fix some notations and terminology. Related to the deformation tensor $\pi_{\alpha\beta}$ of $\mathbf{T}$, we introduce the Σ_t-tangent tensor $h_\alpha^\mu h_\beta^\nu \pi_{\mu\nu}$, where

$$h_\alpha^\beta = \delta_\alpha^\beta + \mathbf{T}_\alpha \mathbf{T}^\beta$$

denotes the projection tensor. It is easy to see that $k_{ij} = h_i^\mu h_j^\nu \pi_{\mu\nu}$ and thus this tensor is an extension of k. We will denote it by the same notation k, i.e.

$$k_{\alpha\beta} = h_\alpha^\mu h_\beta^\nu \pi_{\mu\nu} \tag{4.9}$$

Note that $k_{0\alpha} = k_{\alpha 0} = 0$.

Corresponding to the null vector L, let $\nabla_L k$ be the Σ_t-tangent tensor defined by

$$\nabla_L k_{ij} := h_i^\alpha h_j^\beta \mathbf{D}_L k_{\alpha\beta}$$

and let

$$|\nabla_L k|^2 = g^{ii'} g^{jj'} \nabla_L k_{ij} \nabla_L k_{i'j'}.$$

We also introduce $\not\nabla k$ by $\not\nabla_A k_{ij} := \nabla_A k_{ij}$ and set

$$|\not\nabla k|^2 = \gamma^{AB} g^{ii'} g^{jj'} \nabla_A k_{ij} \nabla_B k_{i'j'}.$$

Corresponding to the second fundamental form k, then, for each $p \in \mathcal{M}_I$, we introduce on the null cone $\mathcal{N}^-(p,\tau)$ the k-flux

$$\mathcal{F}[k](p,\tau) = \int_{\mathcal{N}^-(p,\tau)} \left(|\not\nabla k|^2 + |\nabla_L k|^2 \right), \tag{4.10}$$

where, for each function f and $\tau < i_*(p,t)$,

$$\int_{\mathcal{N}^-(p,\tau)} f := \int_{t(p)-\tau}^{t(p)} \int_{S_t} f n a d\mu_\gamma dt.$$

Corresponding to the time foliation, we recall the null components of the Riemannian curvature tensor $\mathbf{R}$ as follows

$$\begin{aligned} \alpha_{AB} &= \mathbf{R}(L, e_A, L, e_B), & \beta_A &= \frac{1}{2}\mathbf{R}(e_A, L, \underline{L}, L), \\ \rho &= \frac{1}{4}\mathbf{R}(\underline{L}, L, \underline{L}, L), & \sigma &= \frac{1}{4}{}^\star\mathbf{R}(\underline{L}, L, \underline{L}, L), \\ \underline{\beta}_A &= \frac{1}{2}\mathbf{R}(e_A, \underline{L}, \underline{L}, L), & \underline{\alpha}_{AB} &= \mathbf{R}(\underline{L}, e_A, \underline{L}, e_B). \end{aligned} \tag{4.11}$$

The corresponding curvature flux $\mathcal{R}(p,\tau)$ on the null cone $\mathcal{N}^-(p,\tau)$ is given by

$$\mathcal{R}(p,\tau) = \int_{t(p)-\tau}^{t(p)} \int_{S_t} \left(|\alpha|^2 + |\beta|^2 + |\rho|^2 + |\sigma|^2 + |\underline{\beta}|^2\right) nad\mu_\gamma dt.$$

The following result says that once the null lapse a is well controlled, then the k-flux and the curvature flux can be bounded by a universal constant.

Theorem 4.4. *Let the condition* (**A1**) *hold. Then there exists a universal constant $C_* \geq 1$ such that for all $p \in \mathcal{M}_I$ if $|a-1| \leq 1/2$ on $\mathcal{N}^-(p,\tau)$ for some $0 < \tau \leq i_*$ then there holds*

$$\mathcal{R}(p,\tau) + \mathcal{F}[k](p,\tau) \leq C_*.$$

We will prove Theorem 4.4 in Section 6. This result requires $1/2 \leq a \leq 3/2$ on $\mathcal{N}^-(p,\tau)$ which is obvious for small $\tau > 0$ since $a(p) = 1$. In order for the above result to be applicable, we must show that there is a universal constant $\delta_* > 0$ such that the same bound on a holds with $\tau := \min\{i_*, \delta_*\}$, and so does the same bound on $\mathcal{R}(p,\tau) + \mathcal{F}[k](p,\tau)$. We will use a bootstrap argument to achieve this together with various estimates on $\mathrm{tr}\chi$, $\hat{\chi}$ and ν. That is, we will make the following bootstrap assumptions

$$|a-1| \leq \frac{1}{2}, \tag{BA1}$$

$$\left|\mathrm{tr}\chi - \frac{2}{s}\right| \leq \mathcal{E}_0, \tag{BA2}$$

$$\|\hat{\chi}\|^2_{L^\infty_\omega L^2_t(\mathcal{N}^-(p,\tau))} \leq \mathcal{E}_0, \tag{BA3}$$

$$\|\nu\|^2_{L^\infty_\omega L^2_t(\mathcal{N}^-(p,\tau))} \leq \mathcal{E}_0, \tag{BA4}$$

on the null cone $\mathcal{N}^-(p,\tau)$ for all $p \in \mathcal{M}_I$, where $0 < \tau \leq i_*$ and $\mathcal{E}_0 \geq 1$ are two numbers satisfying $\mathcal{E}_0\tau \leq 1$. Due to the continuity of the quantities involved and the compactness of $\mathcal{M}_I$, the bootstrap assumptions (**BA1**)–(**BA4**) hold automatically for sufficiently small $\tau > 0$. Our goal is to show that we can choose universal constants $\mathcal{E}_0 \geq 1$ and $\delta_* > 0$ such that (**BA1**)–(**BA4**) hold with $\tau = \min\{i_*, \delta_*\}$. We will achieve this by showing that the estimates in (**BA1**)–(**BA4**) can be improved.

We will first derive various intermediate consequences of the bootstrap assumptions. In particular, we will derive the estimate on the important quantity $\mathcal{N}_1[\not{\pi}]$ which is defined as follows. For any S_t tangent tensor field F defined on the null cone $\mathcal{N}^-(p,\tau)$, the Sobolev norm $\mathcal{N}_1[F](p,\tau)$ is defined by

$$\mathcal{N}_1[F](p,\tau) := \|r^{-1}F\|_{L^2(\mathcal{N}^-(p,\tau))} + \|\nabla_L F\|_{L^2(\mathcal{N}^-(p,\tau))} + \|\not{\nabla} F\|_{L^2(\mathcal{N}^-(p,\tau))}. \tag{4.12}$$

Recall that the components of the deformation tensor π of $\mathbf{T}$ under transported coordinates are given by $\pi_{00} = 0$, $\pi_{0i} = -n^{-1}\nabla_i n$ and $\pi_{ij} = k_{ij}$. Let us denote by

$\lambda = -\frac{1}{3}\mathrm{Tr}k = -\frac{1}{3}t$ and $\hat{k}$ the traceless part of k. We decompose $\hat{k}$ on each S_t by introducing components

$$\eta_{AB} = \hat{k}_{AB}, \qquad \epsilon_A = \hat{k}_{AN}, \qquad \delta = \hat{k}_{NN} \tag{4.13}$$

where $(e_A)_{A=1,2}$ is an orthonormal frame on S_t and N is the inward unit normal of S_t in Σ_t. Let $\hat{\eta}_{AB}$ be the traceless part of η. Since $\delta^{AB}\eta_{AB} = -\delta$, we have

$$\hat{\eta}_{AB} = \eta_{AB} + \frac{1}{2}\delta_{AB}\delta.$$

We will denote by $\tilde{\not{k}}$, $\not{\nabla}\tilde{\not{k}}$ and $\not{\pi}_0$ the collections

$$\tilde{\not{k}} = (\delta, \epsilon, \hat{\eta}), \qquad \not{\nabla}\tilde{\not{k}} = (\not{\nabla}\delta, \not{\nabla}\epsilon, \not{\nabla}\hat{\eta}), \qquad \not{\pi}_0 = (\not{\nabla}\log n, \nabla_N \log n)$$

respectively. We then define $\not{\pi}$ to be the collection

$$\not{\pi} = (\tilde{\not{k}}, \not{\pi}_0, \lambda). \tag{4.14}$$

We define $\mathcal{N}_1[\not{\pi}](p, \tau)$ according to (4.12) with F replaced by $\not{\pi}$.

With the help of the bound on k-flux given in Theorem 4.4 and various estimates on the lapse n given in Section 3, we will show that $\mathcal{N}_1[\not{\pi}](p, \tau)$ can be bounded in a suitable way under (**A1**) and the bootstrap assumptions.

Theorem 4.5. *Let* (**A1**) *hold. Then there exists a universal constant C such that under the bootstrap assumptions* (**BA1**)–(**BA3**) *with $\mathcal{E}_0\tau \leq 1$ there holds*

$$\mathcal{N}_1[\not{\pi}](p, \tau) \leq C \tag{4.15}$$

for all $p \in \mathcal{M}_I$.

We will prove Theorem 4.5 in Section 8. From Theorem 4.4 and Theorem 4.5 it follows that

$$\mathcal{R}(p, \tau) + \mathcal{N}_1[\not{\pi}](p, \tau) \leq C_0, \tag{4.16}$$

where $C_0 \geq 1$ is a universal constant.

With the help of (4.16), we can establish the following result which enables us to improve the estimates in the bootstrap assumptions.

Theorem 4.6. *There exist two universal constants $\delta_0 > 0$ and $C_1 \geq 1$ such that, under the bootstrap assumptions* (**BA1**)–(**BA4**) *with $\mathcal{E}_0\tau \leq 1$, if $\tau < \min\{i_*, \delta_0\}$ then there hold*

$$|a - 1| \leq C_1\tau^{1/2}, \tag{4.17}$$

$$\left| tr\chi - \frac{2}{s} \right| \leq C_1, \tag{4.18}$$

$$\|\hat{\chi}\|^2_{L^\infty_\omega L^2_t(\mathcal{N}^-(p,\tau))} \leq C_1, \tag{4.19}$$

$$\|\nu\|^2_{L^\infty_\omega L^2_t(\mathcal{N}^-(p,\tau))} \leq C_1 \tag{4.20}$$

on the null cones $\mathcal{N}^-(p, \tau)$ for all $p \in \mathcal{M}_I$.

The significance of Theorem 4.6 lies in that it allows us to choose $\mathcal{E}_0 \geq 1$ and $\delta_* > 0$ universal such that (**BA1**)–(**BA4**) hold on $\mathcal{N}^-(p, \tau)$ with $\tau = \min\{i_*, \delta_*\}$. To see this, we choose $\mathcal{E}_0$ and δ_* in the way that

$$\mathcal{E}_0 := 2C_1 \quad \text{and} \quad \delta_* = \min\{(4C_1)^{-2}, \delta_0\}. \tag{4.21}$$

With such $\mathcal{E}_0$ and δ_*, the estimates (4.17)–(4.20) imply that the estimates (**BA1**)–(**BA4**) can be improved as

$$|a-1| \le \frac{1}{4}, \ \left|\text{tr}\chi - \frac{2}{s}\right| \le \frac{1}{2}\mathcal{E}_0, \ \|\hat{\chi}\|^2_{L^\infty_\omega L^2_t(\mathcal{N}^-(p,\tau))} \le \frac{1}{2}\mathcal{E}_0, \ \|\nu\|^2_{L^\infty_\omega L^2_t(\mathcal{N}^-(p,\tau))} \le \frac{1}{2}\mathcal{E}_0$$

on $\mathcal{N}^-(p,\tau)$ if $\tau \le \min\{i_*, \delta_*\}$. By repeated use of Theorem 4.4, Theorem 4.5 and Theorem 4.6, the bootstrap principle implies that the estimates in the bootstrap assumptions (**BA1**)–(**BA4**) hold with $\tau = \min\{i_*, \delta_*\}$, where $\mathcal{E}_0$ and δ_* are determined by (4.21) which are positive universal constants. Consequently, we obtain (4.4) in Theorem 4.2.

We remark that the analogous results to Theorem 4.6 have been proved in [7, 13] for the geodesic foliations where only the bound of the curvature flux is used. In time foliations, however, the proof of Theorem 4.6 relies not only on the curvature flux but also on $\mathcal{N}_1[\not\pi]$.

Assuming (4.20), the following simple argument shows how to derive (4.17) with the help of (**BA1**). Recall that $a^{-1} = \mathbf{g}(L, \mathbf{T})$ and $L = -a^{-1}(N + \mathbf{T})$. We have

$$\frac{d}{ds}a^{-1} = \mathbf{g}(L, \mathbf{D}_L\mathbf{T}) = a^{-2}\mathbf{g}(N, \mathbf{D_T T}) + a^{-2}\mathbf{g}(N, \mathbf{D}_N\mathbf{T}).$$

Since $\mathbf{D_T T} = n^{-1}\nabla n$ and $k_{NN} = -\langle N, \mathbf{D}_N \mathbf{T}\rangle$ we obtain $\frac{d}{ds}a^{-1} = -a^{-2}\left(\pi_{0N} + k_{NN}\right)$. Consequently

$$L(a) = \frac{d}{ds}a = \pi_{0N} + k_{NN}. \tag{4.22}$$

Since $\frac{ds}{dt} = -na$, we have

$$\frac{d}{dt}a = -na\left(\pi_{0N} + k_{NN}\right).$$

Integrating the above equation along null geodesics initiating from p and using $a(p) = 1$ yields

$$a - 1 = \int_t^{t(p)} \left(\pi_{0N} + k_{NN}\right) nadt' = \int_t^{t(p)} \nu nadt'.$$

Since $0 < n \le 3/t_*^2$, (**BA1**) and (4.20) imply

$$|a-1| \le C_1(t(p)-t)^{1/2} \le C_1\tau^{1/2}$$

for all $t(p) - \tau \le t \le t(p)$, where C_1 could be a different but universal constant.

The derivation of (4.18)–(4.20) however is highly nontrivial and requires lengthy calculation. The complete proof is contained in [14] where other related estimates for Ricci coefficients are proved simultaneously.

In order to complete the proof of Theorem 4.2, it remains to prove (4.3) which is restated in the following result.

Theorem 4.7. *Assume that the condition* (**A1**) *holds. Then there exist universal constants* $\delta_* > 0$ *and* $C > 0$ *such that*

$$\int_{t(p)-\min\{i_*,\delta_*\}}^{t(p)} |k(\Phi(t))|^2 ndt \le C$$

for all $p \in \mathcal{M}_I$, *where* Φ *denotes the integral curve of* $\mathbf{T}$ *through* p.

The proof of Theorem 4.7 forms the core part of the present paper. It is based on the formula of $\Box k$ given in Section 5 and a Kirchoff-Sobolev representation for k given in Section 9 together with various estimates on null cones derived in Section 8.

5. Tensorial wave equation for the second fundamental form

In this section we will derive the formula for $\Box k$, where k is defined in (4.9) whose projection to Σ_t is exactly the second fundamental form.

Proposition 8. *The tensor k defined by (4.9) verifies the tensorial wave equation*

$$\begin{aligned}\Box k_{ij} &= -n^{-3}\dot{n}\nabla_i\nabla_j n + n^{-2}\nabla_i\nabla_j\dot{n} + 2\pi_{0a}\left(\nabla^a k_{ij} - \nabla_i k_j^a - \nabla_j k_i^a\right)\\ &\quad - 2\,Trk\,R_{ij} - Rk_{ij} + R\,Trk\,g_{ij} + 2(k_i^a R_{aj} + k_j^a R_{ai}) - 2R_{ab}k^{ab}g_{ij}\\ &\quad + n^{-1}(2k_i^a\nabla_a\nabla_j n + 2k_j^a\nabla_a\nabla_i n - \Delta n k_{ij} - Trk\,\nabla_i\nabla_j n)\\ &\quad + 2k_{ia}k^{ab}k_{bj} - \pi_{0a}\pi_0^a k_{ij} - n^{-1}k_{ij}. \end{aligned} \tag{5.1}$$

Proof. We first recall that

$$\Box k_{ij} = -\mathbf{D}_0\mathbf{D}_0 k_{ij} + g^{pq}\mathbf{D}_p\mathbf{D}_q k_{ij}.$$

By using $k_{0\alpha} = k_{\alpha 0} = 0$ and $\mathbf{D}_i e_j = \nabla_i e_j - k_{ij}\mathbf{T}$, we can obtain through a straightforward calculation that

$$g^{pq}\mathbf{D}_p\mathbf{D}_q k_{ij} = \triangle k_{ij} + \mathrm{Tr}k\,\mathbf{D}_0 k_{ij} + 2k_{ia}k^{ab}k_{bj}.$$

By using $\mathbf{D_T T} = n^{-1}\nabla^i n e_i = -\pi_0^i e_i$ and $k_{0\alpha} = k_{\alpha 0} = 0$, we can obtain

$$\begin{aligned}\mathbf{D}_0\mathbf{D}_0 k_{ij} &= e_0(\mathbf{D}_0 k_{ij}) + k_i^a\mathbf{D}_0 k_{aj} + k_j^a\mathbf{D}_0 k_{ia} + \pi_{0a}\nabla^a k_{ij}\\ &\quad + \pi_{0i}\mathbf{D}_0 k_{0j} + \pi_{0j}\mathbf{D}_0 k_{i0}.\end{aligned}$$

It is easy to see $\mathbf{D}_0 k_{0j} = \pi_{0a}k_j^a$. From the equation (1.3) it also follows that

$$\mathbf{D}_0 k_{ij} = e_0(k_{ij}) + 2k_{ia}k_j^a = -n^{-1}\nabla_i\nabla_j n + R_{ij} + \mathrm{Tr}k\,k_{ij}. \tag{5.2}$$

Consequently

$$\begin{aligned}\mathbf{D}_0\mathbf{D}_0 k_{ij} &= e_0(\mathbf{D}_0 k_{ij}) + \pi_{0a}\nabla^a k_{ij} - n^{-1}\left(k_i^a\nabla_a\nabla_j n + k_j^a\nabla_a\nabla_i n\right)\\ &\quad + \left(k_i^a R_{aj} + k_j^a R_{ai}\right) + 2\mathrm{Tr}k\,k_{ia}k_j^a + \pi_{0i}\pi_{0a}k_j^a + \pi_{0j}\pi_{0a}k_i^a.\end{aligned}$$

Therefore

$$\begin{aligned}\Box k_{ij} &= -e_0(\mathbf{D}_0 k_{ij}) - \pi_{0a}\nabla^a k_{ij} - \pi_{0i}\pi_{0a}k_j^a - \pi_{0j}\pi_{0a}k_i^a\\ &\quad + n^{-1}\left(k_i^a\nabla_a\nabla_j n + k_j^a\nabla_a\nabla_i n\right) - \left(k_i^a R_{aj} + k_j^a R_{ai}\right)\\ &\quad - 2\mathrm{Tr}k\,k_{ia}k_j^a + \triangle k_{ij} + \mathrm{Tr}k\,\mathbf{D}_0 k_{ij} + 2k_{ia}k^{ab}k_{bj}.\end{aligned} \tag{5.3}$$

We need to compute $e_0(\mathbf{D}_0 k_{ij})$. It follows from (5.2) and $\mathrm{Tr}k = t$, we have

$$\begin{aligned}e_0(\mathbf{D}_0 k_{ij}) &= n^{-3}\dot{n}\nabla_i\nabla_j n - n^{-2}\partial_t(\nabla_i\nabla_j n) + n^{-1}\partial_t R_{ij}\\ &\quad + n^{-1}k_{ij} + \mathrm{Tr}k\,\mathbf{D}_0 k_{ij} - 2\mathrm{Tr}k\,k_{ia}k_j^a.\end{aligned} \tag{5.4}$$

In order to compute $\partial_t(\nabla_i\nabla_j n)$ and $\partial_t R_{ij}$, let Γ_{ij}^a denote the Christoffel symbol of Σ_t. Then it follows from the equation $\partial_t g_{ij} = -2nk_{ij}$ that

$$\dot{\Gamma}_{ij}^a = -n\left(\nabla_i k_j^a + \nabla_j k_i^a - \nabla^a k_{ij}\right) - \nabla_i n k_j^a - \nabla_j n k_i^a + \nabla^a n k_{ij}.$$

Using $\text{div}k = 0$ and $\text{Tr}k = t$, this in particular implies $\dot{\Gamma}^a_{aj} = -\text{Tr}k\nabla_j n$. Therefore, noting that $\partial_t(\nabla_i\nabla_j n) = \nabla_i\nabla_j\dot{n} - \dot{\Gamma}^a_{ij}\nabla_a n$, we can obtain

$$\begin{aligned}\partial_t(\nabla_i\nabla_j n) &= \nabla_i\nabla_j\dot{n} + n\nabla_a n\left(\nabla_i k^a_j + \nabla_j k^a_i - \nabla^a k_{ij}\right)\\ &\quad + \left(\nabla_i n k^a_j + \nabla_j n k^a_i\right)\nabla_a n - |\nabla n|^2 k_{ij}.\end{aligned} \tag{5.5}$$

Noting also that $\partial_t R_{ij} = \nabla_a\dot{\Gamma}^a_{ij} - \nabla_i\dot{\Gamma}^a_{aj}$ and $\text{div}k = 0$, we have

$$\begin{aligned}\partial_t R_{ij} &= \nabla_a n\left(2\nabla^a k_{ij} - \nabla_i k^a_j - \nabla_j k^a_i\right) - n\left(\nabla_a\nabla_i k^a_j + \nabla_a\nabla_j k^a_i - \triangle k_{ij}\right)\\ &\quad + \Delta n k_{ij} - \left(\nabla_a\nabla_i n\cdot k^a_j + \nabla_a\nabla_j n\cdot k^a_i\right) + \text{Tr}k\,\nabla_i\nabla_j n.\end{aligned}$$

With the help of the commutation formula

$$\nabla_a\nabla_i k^a_j = [\nabla_a,\nabla_i]k^a_j = R_j{}^a{}_{bi}k^b_a + R_{ai}k^a_j$$

and the curvature decomposition formula

$$R_j{}^a{}_{bi} = g_{jb}R^a_i + R_{jb}\delta^a_i - R_{ij}\delta^a_b - R^a_b g_{ji} - \frac{1}{2}(g_{jb}\delta^a_i - g_{ij}\delta^a_b)R,$$

we obtain

$$\nabla_a\nabla_i k^a_j = 2R_{ia}k^a_j + R_{ja}k^a_i - \text{Tr}k\,R_{ij} - R_{ab}k^{ab}g_{ij} - \frac{1}{2}Rk_{ij} + \frac{1}{2}R\,\text{Tr}k\,g_{ij}.$$

Consequently

$$\begin{aligned}\partial_t R_{ij} &= \nabla_a n\left(2\nabla^a k_{ij} - \nabla_i k^a_j - \nabla_j k^a_i\right) - \left(\nabla_a\nabla_i n k^a_j + \nabla_a\nabla_j n k^a_i\right)\\ &\quad + n\triangle k_{ij} + \triangle n k_{ij} - 3n\left(R_{ia}k^a_j + R_{ja}k^a_i\right) + 2n\text{Tr}k\,R_{ij}\\ &\quad + 2nR_{ab}k^{ab}g_{ij} + nRk_{ij} - nR\text{Tr}k\,g_{ij} + \text{Tr}k\,\nabla_i\nabla_j n.\end{aligned} \tag{5.6}$$

Plugging (5.5) and (5.6) into (5.4), and using $\pi_{0i} = -n^{-1}\nabla_i n$, it yields

$$\begin{aligned}e_0(\mathbf{D}_0 k_{ij}) &= n^{-3}\dot{n}\nabla_i\nabla_j n - n^{-2}\nabla_i\nabla_j\dot{n} - \pi_{0a}\left(3\nabla^a k_{ij} - 2\nabla_i k^a_j - 2\nabla_j k^a_i\right) - \pi_{0i}\pi_{0a}k^a_j\\ &\quad - \pi_{0j}\pi_{0a}k^a_i + \pi_{0a}\pi^a_0 k_{ij} - n^{-1}\left(\nabla_a\nabla_i n k^a_j + \nabla_a\nabla_j n k^a_i - \text{Tr}k\,\nabla_i\nabla_j n\right)\\ &\quad + \triangle k_{ij} + n^{-1}\triangle n\,k_{ij} - 3\left(R_{ia}k^a_j + R_{ja}k^a_i\right) + 2\text{Tr}k\,R_{ij} + 2R_{ab}k^{ab}g_{ij}\\ &\quad + Rk_{ij} - R\text{Tr}k\,g_{ij} + n^{-1}k_{ij} + \text{Tr}k\,\mathbf{D}_0 k_{ij} - 2\text{Tr}k\,k^a_i k_{aj}.\end{aligned}$$

Plugging the above equation into (5.3) gives the desired equation. □

6. **Proof of Theorem 4.4**

In this section we will complete the proof of Theorem 4.4, i.e. we will show that if $|a-1| \le 1/2$ on $\mathcal{N}^-(p,\tau)$ for some $0 < \tau \le i_*$ then

$$\mathcal{R}(p,\tau) + \mathcal{F}[k](p,\tau) \le C_*,$$

where C_* is a universal constant.

We will use the following result (see [5, Lemma 8.1.1]).

Lemma 6.1. *Let P be a vector field defined on the domain $\mathcal{J}^-(p,\tau)$. Then*

$$\int_{\mathcal{N}^-(p,\tau)} \mathbf{g}(P,L) = \int_{\mathcal{J}^-(p,\tau)} \mathbf{D}^\mu P_\mu - \int_{\Sigma_{t(p)-\tau}\cap\mathcal{J}^-(p)} \mathbf{g}(P,\mathbf{T})d\mu_g,$$

where $\mathcal{J}^-(p)$ denotes the causal past of p, $\mathcal{J}^-(p,\tau)$ denotes the portion of $\mathcal{J}^-(p)$ in the slab $[t(p)-\tau, t(p)]$, and

$$\int_{\mathcal{J}^-(p,\tau)} f = \int_{t(p)-\tau}^{t(p)} dt \left(\int_{\Sigma_t \cap \mathcal{J}^-(p)} f n d\mu_g \right).$$

We first show the boundedness of the curvature flux $\mathcal{R}(p,\tau)$. With the Bel-Robinson tensor $\mathbf{Q}[\mathbf{R}]$ defined in Section 2, we introduce $P_\mu = Q[\mathbf{R}]_{\mu\beta\gamma\delta}\mathbf{T}^\beta\mathbf{T}^\gamma\mathbf{T}^\delta$. We may apply Lemma 6.1 to obtain

$$\int_{\mathcal{N}^-(p,\tau)} \mathbf{g}(P,L) = \int_{\mathcal{J}^-(p,\tau)} \mathbf{D}^\mu P_\mu - \int_{\Sigma_{t(p)-\tau}\cap \mathcal{J}^-(p)} \mathbf{Q}[\mathbf{R}](\mathbf{T},\mathbf{T},\mathbf{T}.\mathbf{T}) d\mu_g,$$

With the help of the calculations in subsection 2.1, (**A1**) and Lemma 2.1, it then yields

$$\left| \int_{\mathcal{N}^-(p,\tau)} \mathbf{g}(P,L) \right| \lesssim C. \tag{6.1}$$

Note that $\mathbf{g}(P,L) = \mathbf{Q}[\mathbf{R}](\mathbf{T},\mathbf{T},\mathbf{T},L)$ and $\mathbf{T} = -\frac{1}{2}(aL + a^{-1}\underline{L})$. Since $|a-1| \le 1/2$ on $\mathcal{N}^-(p,\tau)$, it follows from [5, Lemma 7.3.1] that $-\mathbf{g}(P,L)$ is equivalent to

$$|\alpha|^2 + |\beta|^2 + |\underline{\beta}|^2 + |\rho|^2 + |\sigma|^2.$$

Thus, there holds, for some universal constant $C > 0$,

$$C^{-1}\mathcal{R}(p,\tau) \le \left| \int_{\mathcal{N}^-(p,\tau)} \mathbf{g}(P,L) \right| \le C\mathcal{R}(p,\tau).$$

By (6.1), we conclude that $\mathcal{R}(p,\tau) \le C_*$ for some universal constant C_*.

Next we will show the boundedness of the k-flux $\mathcal{F}[k](p,\tau)$. With the help of the projection tensor

$$h^{\alpha\beta} = \mathbf{g}^{\alpha\beta} + \mathbf{T}^\alpha\mathbf{T}^\beta,$$

for any tensor field $U_{\alpha_1\alpha_2\cdots\alpha_m}$ in $T\mathcal{M}$, we define $|U|$ as follows

$$\begin{aligned} |U|^2 &= h^{IJ}U_I U_J = h^{\alpha_1\beta_1}\cdots h^{\alpha_m\beta_m} U_{\alpha_1\alpha_2\cdots\alpha_m} U_{\beta_1\beta_2\cdots\beta_m} \\ h^{IJ} &= h^{\alpha_1\beta_1}\cdots h^{\alpha_m\beta_m}, \quad U_I = U_{\alpha_1\alpha_2\cdots\alpha_m}, \quad U_J = U_{\beta_1\beta_2\cdots\beta_m}. \end{aligned}$$

For any Σ_t-tangent tensor field U in $\mathcal{M}_I$, we define the energy momentum tensor $Q[U]_{\alpha\beta}$ associated with the covariant wave operator acting on tensors:

$$Q[U]_{\alpha\beta} := h^{IJ}\mathbf{D}_\alpha U_I \mathbf{D}_\beta U_J - \frac{1}{2}\mathbf{g}_{\alpha\beta} h^{IJ}\mathbf{g}^{\mu\nu}\mathbf{D}_\mu U_I \mathbf{D}_\nu U_J.$$

We have

$$\begin{aligned} \mathbf{D}^\beta Q[U]_{\alpha\beta} &= h^{IJ}\mathbf{D}_\alpha U_I \Box U_J + h^{IJ}(\mathbf{D}_\beta\mathbf{D}_\alpha U_I - \mathbf{D}_\alpha\mathbf{D}_\beta U_I)\mathbf{D}^\beta U_J \\ &\quad + \mathbf{D}^\beta h^{IJ}(\mathbf{D}_\alpha U_I \mathbf{D}_\beta U_J - \frac{1}{2}\mathbf{g}_{\alpha\beta}\mathbf{g}^{\mu\nu}\mathbf{D}_\mu U_I \mathbf{D}_\nu U_J) \end{aligned}$$

It is easy to see that the last term in the above equation can be written symbolically as $\pi \cdot \mathbf{D}U \cdot \mathbf{D}U$.

Now we apply the above equation to $U = k$. Noting that $h^{0\alpha} = 0$ and $h^{ij} = g^{ij}$, we have

$$\begin{aligned}\mathbf{D}^\beta(Q[k]_{\alpha\beta}\mathbf{T}^\alpha) &= \mathbf{D}^\beta\mathbf{T}^\alpha Q[k]_{\alpha\beta} + \mathbf{D}^\beta Q[k]_{0\beta}\\ &= -k^{ij}Q[k]_{ij} - \pi^{0j}Q[k]_{0j} + \mathbf{D}_0 k^{ij}\Box k_{ij}\\ &\quad + [\mathbf{D}_a, \mathbf{D}_0]k_{ij}\nabla^a k^{ij} + \pi\cdot\mathbf{D}k\cdot\mathbf{D}k.\end{aligned} \tag{6.2}$$

In view of the commutation formula

$$[\mathbf{D}_m, \mathbf{D}_0]k_{ij} = \mathbf{R}_i{}^b{}_{m0}k_{bj} + \mathbf{R}_j{}^b{}_{m0}k_{ib} = -\epsilon_{ib}^s H_{sm}k_j^b - \epsilon_{jb}^s H_{sm}k_i^b,$$

we derive symbolically

$$\begin{aligned}\mathbf{D}^\beta(Q[k]_{\alpha\beta}\mathbf{T}^\alpha) &= -k^{ij}Q[k]_{ij} - \pi^{0j}Q[k]_{0j} + \mathbf{D}_0 k^{ij}\Box k_{ij}\\ &\quad + H\cdot k\cdot\nabla k + \pi\cdot\mathbf{D}k\cdot\mathbf{D}k.\end{aligned} \tag{6.3}$$

From the definition of $Q[k]$, it is easy to see that

$$Q[k]_{00} = \frac{1}{2}(|\mathbf{D}_0 k|^2 + |\nabla k|^2), \tag{6.4}$$

$$Q[k]_{0j} = \mathbf{D}_0 k_{pq}\nabla_j k^{pq}, \tag{6.5}$$

$$Q[k]_{ij} = \nabla_i k_{pq}\nabla_j k^{pq} - \frac{1}{2}g_{ij}\left(-|\mathbf{D}_0 k|^2 + |\nabla k|^2\right). \tag{6.6}$$

Therefore

$$\begin{aligned}\mathbf{D}^\beta(Q[k]_{\alpha\beta}\mathbf{T}^\alpha) &= \frac{1}{2}\mathrm{Tr}k\left(-|\mathbf{D}_0 k|^2 + |\nabla k|^2\right) + k\cdot\nabla k\cdot\nabla k\\ &\quad + \mathbf{D}_0 k\cdot\Box k + H\cdot k\cdot\nabla k + \pi\cdot\mathbf{D}k\cdot\mathbf{D}k.\end{aligned} \tag{6.7}$$

We now apply Lemma 6.1 to $P^\beta := \mathbf{T}^\alpha Q[k]_\alpha^\beta$ and obtain

$$\int_{\mathcal{N}^-(p,\tau)} Q[k](\mathbf{T}, L) + \int_{\Sigma_{t(p)-\tau}\cap\mathcal{J}^-(p)} Q[k]_{00} = \int_{\mathcal{J}^-(p,\tau)} \mathbf{D}^\beta(Q[k]_{\alpha\beta}\mathbf{T}^\alpha). \tag{6.8}$$

For the null pair L and $\underline{L}$, it is easy to see that

$$Q[k](L,L) = |\nabla_L k|^2, \qquad Q[k](\underline{L}, L) = |\nabla\!\!\!/ k|^2.$$

Since $\mathbf{T} = -\frac{1}{2}(aL + a^{-1}\underline{L})$, we have

$$Q[k](\mathbf{T}, L) = -\frac{1}{2}\left(aQ[k](L,L) + a^{-1}Q[k](\underline{L}, L)\right) = -\frac{1}{2}\left(a|\nabla_L k|^2 + a^{-1}|\nabla\!\!\!/ k|^2\right).$$

Since $|a-1| \le 1/2$, the k-flux defined in (4.10) verifies the inequality

$$-\int_{\mathcal{N}^-(p,\tau)} Q[k](\mathbf{T}, L) \le \mathcal{F}[k](p,\tau) \le -4\int_{\mathcal{N}^-(p,\tau)} Q[k](\mathbf{T}, L).$$

Thus we derive from (6.8) and (6.4) that

$$\mathcal{F}[k](p,\tau) \le 4\left|\int_{\mathcal{J}^-(p,\tau)} \mathbf{D}^\beta(Q[k]_{\alpha\beta}\mathbf{T}^\alpha)\right| + 2\int_{\Sigma_{t(p)-\tau}\cap\mathcal{J}^-(p)} \left(|\mathbf{D}_0 k|^2 + |\nabla k|^2\right). \tag{6.9}$$

In view of (5.2), Lemma 2.2, Proposition 3 and Proposition 6, we have

$$\begin{aligned}&\int_{\Sigma_t}\left(|\mathbf{D}_0 k|^2 + |\nabla k|^2\right)\\ &\lesssim \|\nabla^2 n\|^2_{L^2(\Sigma_t)} + \|Ric\|^2_{L^2(\Sigma_t)} + \|k\|^4_{L^4(\Sigma_t)} + \|\nabla k\|^2_{L^2(\Sigma_t)} \le C.\end{aligned} \tag{6.10}$$

Moreover, in view of (6.7), (**A1**), Lemma 2.2, and the above inequality we have

$$\begin{aligned}\left|\int_{\mathcal{J}^-(p,\tau)} \mathbf{D}^\beta(Q[k]_{\alpha\beta}\mathbf{T}^\alpha)\right| &\lesssim \int_{t(p)-\tau}^{t(p)} \|\mathbf{D}_0 k\|_{L^2(\Sigma_{t'})}\|\Box k\|_{L^2(\Sigma_{t'})}dt' \\ &\quad + \int_{t(p)-\tau}^{t(p)} \|\pi\|_{L^\infty(\Sigma_{t'})}\left(\|\mathbf{D}_0 k\|^2_{L^2(\Sigma_{t'})} + \|\nabla k\|^2_{L^2(\Sigma_{t'})}\right)dt' \\ &\quad + \int_{t(p)-\tau}^{t(p)} \|k\|_{L^\infty(\Sigma_{t'})}\|H\|_{L^2(\Sigma_{t'})}\|\nabla k\|_{L^2(\Sigma_{t'})}dt' \\ &\lesssim \int_{t(p)-\tau}^{t(p)} \|\Box k\|_{L^2(\Sigma_{t'})}dt' + \int_{t(p)-\tau}^{t(p)} \|\pi\|_{L^\infty(\Sigma_{t'})}dt' \\ &\le C + C\int_{t(p)-\tau}^{t(p)} \|\Box k\|_{L^2(\Sigma_{t'})}dt'. \qquad (6.11)\end{aligned}$$

Therefore

$$\mathcal{F}[k](p,\tau) \le C + C\int_{t(p)-\tau}^{t(p)} \|\Box k\|_{L^2(\Sigma_{t'})}dt'. \qquad (6.12)$$

We now recall the formula for $\Box k$ given in Proposition 8 which symbolically can be written as

$$\Box k = -n^{-3}\dot{n}\nabla^2 n + n^{-2}\nabla^2\dot{n} + \pi\cdot\pi\cdot\pi + k\cdot\nabla^2 n + k\cdot Ric + \pi\cdot\nabla k - n^{-1}k.$$

Since $C^{-1} \le n \le C$, we obtain

$$\begin{aligned}\|\Box k\|_{L^1_t L^2_x} &\lesssim \|\dot{n}\|_{L^1_t L^\infty_x}\|\nabla^2 n\|_{L^\infty_t L^2_x} + \|\nabla^2\dot{n}\|_{L^1_t L^2_x} + \|\pi\|_{L^1_t L^\infty_x}\|\pi\|^2_{L^\infty_t L^4_x} \\ &\quad + \|k\|_{L^1_t L^\infty_x}\|\nabla^2 n\|_{L^\infty_t L^2_x} + \|k\|_{L^1_t L^\infty_x}\|Ric\|_{L^\infty_t L^2_x} \\ &\quad + \|k\|_{L^1_t L^2_x} + \|\pi\|_{L^1_t L^\infty_x}\|\nabla k\|_{L^\infty_t L^2_x}.\end{aligned}$$

In view of the assumption (**A1**), Lemma 2.2, Proposition 3, Proposition 7 and (6.10), it follows

$$\|\Box k\|_{L^1_t L^2_x} \le C\left(1 + \|\pi\|_{L^1_t L^\infty_x} + \tau\right) \le C.$$

Combining the above inequality with (6.12) completes the proof of Theorem 4.4.

7. **Trace estimates**

For a point $p \in \mathcal{M}_I$, let s be the affine parameter on the null cone $\mathcal{N}^-(p)$ and let r be the radius of $S_t := \mathcal{N}^-(p) \cap \Sigma_t$ which is defined by (4.2). On each S_t we introduce the ratio of area elements

$$v_t(\omega) = \frac{\sqrt{|\gamma|}}{\sqrt{|\mathring{\gamma}|}}, \qquad \omega \in \mathbb{S}^2. \qquad (7.1)$$

We will first show that all the quantities s, r, $v_t^{1/2}$ and $t(p)-t$ are comparable under the bootstrap assumptions (**BA1**)–(**BA3**). Here we say two quantities φ and ψ are comparable in the sense that $C^{-1}\psi \le \varphi \le C\psi$ for some universal constant $C > 0$.

Lemma 7.1. *Under the bootstrap assumptions* (**BA1**)–(**BA3**)*, the four quantities* $s(t)$, $r(t)$, $v_t^{1/2}$ *and* $t(p)-t$ *are comparable on the null cone* $\mathcal{N}^-(p,\tau)$ *with* $\tau \le \min\{i_*, \delta_*\}$*, where* $\delta_* > 0$ *is a universal constant.*

Proof. The comparability of s and $t(p)-t$ follows from the relation $\frac{ds}{dt}=-na$ and the bootstrap assumption (**BA1**). Similar to the derivation of (4.5), we have under the bootstrap assumptions (**BA1**)–(**BA3**) that

$$|\gamma - s(t)^2 \overset{\circ}{\gamma}| \le \frac{1}{2} s(t)^2 \overset{\circ}{\gamma} \tag{7.2}$$

for all $t(p)-\min\{i_*,\tau,\delta_*\} \le t < t(p)$, where δ_* is a universal constant. This implies immediately that $\frac{1}{2}s(t)^2 \le v_t \le \frac{3}{2}s(t)^2$. Consequently v_t and $t(p)-t$ are comparable. Thus for the area $|S_t|$ of S_t there holds

$$C^{-1}(t(p)-t)^2 \le |S_t| \le C(t(p)-t)^2$$

for some universal constant C. This together with the definition of r gives the comparability of r and $t(p)-t$. □

7.1. **Optical function.** In this section we give a brief review of the construction of optical functions, one may see [5] for more information.

For any point $p \in \mathcal{M}_I$, let $J^-(p)$ be the causal past and let $\mathcal{N}^-(p)$ and $\mathcal{I}^-(p)$ denote respectively the null boundary and the interior. For each $0<\tau<i_*$ with i_* defined by (4.1), let $\mathcal{J}^-(p,\tau)$, $\mathcal{N}^-(p,\tau)$ and $\mathcal{I}^-(p,\tau)$ denote the portions of $\mathcal{J}^-(p)$, $\mathcal{N}^-(p)$ and $\mathcal{I}^-(p)$ in the time slab $[t(p)-\tau, t(p)]$ respectively. Let Φ be the integral curve of $\mathbf{T}$ through p with $\Phi(t(p))=p$. According to the definition of i_*, all the null cones $\mathcal{N}^-(\Phi(t), \tau+t-t(p))$, with $t(p)-\tau \le t \le t(p)$ and $\tau < i_*$, are disjoint and their union forms $\mathcal{N}^-(p,\tau)$. We now define u to be the function, constant on each $\mathcal{N}^-(\Phi(t), t+\tau-t(p))$, such that

$$u(\Phi(t)) = \int_{t_0}^{t} n(\Phi(t'))dt'.$$

Such u, which will be called an optical function, is a well-defined smooth function on $\mathcal{J}^-(p,\tau)$ and satisfies the eikonal equation

$$\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0.$$

It is clear that the level sets C_u of u are the incoming null cones in the time slab $[t(p)-\tau, t(p)]$ with vertices on Φ, and $\mathbf{T}(u)=1$ on Φ. Moreover, the null geodesic vector L defined before can be written as

$$L = \mathbf{g}^{\alpha\beta}\partial_\beta u \partial_\alpha.$$

For each $t \in [t(p)-\tau, t(p)]$, we define $u_M(t)$ and $u_m(t)$ respectively to be the largest and smallest values of u for which the part of the cone C_u that lies in the future of Σ_t is contained in $\mathcal{J}^-(p)$, i.e.

$$u_M(t) = u(p) \qquad \text{and} \qquad u_m(t) = u(\Phi(t)).$$

For each $u(\Phi(t(p)-\tau)) \le u \le u(p)$, we also define $t_M(u)$ and $t_m(u)$ to be the largest and smallest value of t for which Σ_t intersects C_u respectively. It is clear that $t_M(u)$ is the value of t at the vertex of C_u and $t_m(u) = t(p)-\tau$. Note that both u_M and t_m are independent of t.

We set

$$S_{t,u} := C_u \cap \Sigma_t$$

which is a smooth surface for each $t(p)-\tau \le t \le t(p)$ and $u_M \le u < u_m(t)$. The corresponding radius function is defined as

$$r(t,u) := \sqrt{(4\pi)^{-1}|S_{t,u}|},$$

where $|S_{t,u}|$ denotes the area of $S_{t,u}$ with respect to the metric γ.

The following result follows immediately from Lemma 7.1 and the definition of u.

Proposition 9. *Under the bootstrap assumptions* (**BA1**)–(**BA3**) *on* $\mathcal{N}^-(p,\tau)$ *for all* $p \in \mathcal{M}_I$, *there hold*

$$C^{-1} \le \frac{t_M(u)-t}{r(t,u)} \le C \tag{7.3}$$

and

$$C^{-1} \le \frac{u-u_m(t)}{r(t,u)} \le C \tag{7.4}$$

for all $t(p)-\min\{i_*,\tau,\delta_*\} < t < t(p)$, *where* C *and* δ_* *are two positive universal constants.*

In view of the above notations, it is clear that

$$\mathcal{N}^-(p,\tau) = \bigcup_{t\in[t(p)-\tau,t(p)]} S_{t,u_M}.$$

Let $\mathrm{Int}(S_{t,u_M})$ be the interior of S_{t,u_M} in Σ_t, then

$$\mathrm{Int}(S_{t,u_M}) = \bigcup_{u\in[u_M,u_m(t)]} S_{t,u} \quad \text{and} \quad \mathcal{J}^-(p,\tau) = \bigcup_{t\in[t(p)-\tau,t(p)]} \mathrm{Int}(S_{t,u_M}).$$

The following simple result can be found in [5].

Lemma 7.2. *For any scalar* f *satisfying*

$$\lim_{u\to u_m(t)} \int_{S_{t,u}} f d\mu_\gamma = 0,$$

there holds

$$\int_{S_{t,u_M}} f d\mu_\gamma = -\int_{u_m(t)}^{u_M}\int_{S_{t,u}} (\nabla_N f + tr\theta f) a d\mu_{\gamma_u} du,$$

where N *denotes the unit inward normal to* $S_{t,u}$ *in* Σ_t, *and* θ *denotes the corresponding second fundamental form.*

7.2. **Trace estimates.** We will rely on the following trace inequality.

Lemma 7.3 (Trace inequality)**.** *Under the bootstrap assumptions* (**BA1**)–(**BA3**) *on* $\mathcal{N}^-(p,\tau)$ *with* $\mathcal{E}_0\tau \le 1$, *for any* Σ_t *tangent tensor field* F *there holds*

$$\|r^{-1/2}F\|_{L^2(S_t)} \lesssim \|\nabla F\|_{L^2(\Sigma_t)} + \|F\|_{L^2(\Sigma_t)},$$

where $S_t := \mathcal{N}^-(p,\tau)\cap\Sigma_t$ *and* $r := \sqrt{(4\pi)^{-1}|S_t|}$.

The proof of Lemma 7.3 can be seen in Appendix. Using Lemma 7.3, we are able to derive the following

Proposition 10. *Let the bootstrap assumptions* (**BA1**)–(**BA3**) *hold on* $\mathcal{N}^-(p,\tau)$ *with* $\mathcal{E}_0\tau \le 1$. *Then for any* Σ_t *tangent tensor field* F *there hold*

$$\|F\|^2_{L^2(S_t)} \lesssim \|F\|_{H^1(\Sigma_t)}\|F\|_{L^2(\Sigma_t)}, \tag{7.5}$$

$$\|F\|_{L^4(S_t)} \lesssim \|F\|_{H^1(\Sigma_t)} \tag{7.6}$$

for all $t(p)-\tau \le t < t(p)$.

Proof. Let $\phi(u)$ be a smooth cut-off function verifying $0 \le \phi \le 1$, $\phi(u_M) = 1$ and $\mathrm{supp}(\phi) \subset [\frac{u_m+u_M}{2}, u_M]$. It then follows from Lemma 7.2 that

$$\|F\|^2_{L^2(S_t)} = -\int_{\mathrm{Int}(S_t)} \left(\nabla_N |\phi F|^2 + \mathrm{tr}\theta |\phi F|^2\right) a d\mu_\gamma du' = I_1 + I_2, \tag{7.7}$$

where

$$I_1 = -2\int_{\mathrm{Int}(S_t)} \left(\phi^2 F \cdot \nabla_N F + \phi \nabla_N \phi |F|^2\right) a d\mu_\gamma du',$$
$$I_2 = -\int_{\mathrm{Int}(S_t)} \mathrm{tr}\theta |\phi F|^2 a d\mu_\gamma du'.$$

Since the bootstrap assumption (**BA1**) implies $1/2 \le a \le 3/2$, it is easy to see that

$$\left|\int_{\mathrm{Int}(S_t)} \phi^2 F \cdot \nabla_N F a d\mu_\gamma du'\right| \lesssim \|\nabla_N F\|_{L^2(\Sigma_t)} \|F\|_{L^2(\Sigma_t)}$$

and

$$\left|\int_{\mathrm{Int}(S_t)} \phi \nabla_N \phi |F|^2 a d\mu_\gamma du'\right| \lesssim \frac{1}{u_M - u_m} \int_{\frac{u_m+u_M}{2}}^{u_M} \int_{S_{t,u'}} |F|^2 d\mu_\gamma du'.$$

It follows from Lemma 7.3 that

$$\int_{S_{t,u'}} |F|^2 d\mu_\gamma \lesssim \|r^{-1/2} F\|_{L^2(S_{t,u'})} \|F\|_{L^2(S_{t,u'})} r^{1/2}$$
$$\lesssim \|F\|_{H^1(\Sigma_t)} \|F\|_{L^2(S_{t,u'})} r^{1/2},$$

where $r := r(t, u')$. From Proposition 9 it follows that $r(t, u') \lesssim u' - u_m$. Thus

$$\left|\int_{\mathrm{Int}(S_t)} \phi \nabla_N \phi |F|^2 a d\mu_\gamma du'\right|$$
$$\lesssim \frac{1}{u_M - u_m} \|F\|_{H^1(\Sigma_t)} \|F\|_{L^2(\Sigma_t)} \left(\int_{\frac{u_m+u_M}{2}}^{u_M} (u' - u_m) du'\right)^{1/2}$$
$$\lesssim \|F\|_{H^1(\Sigma_t)} \|F\|_{L^2(\Sigma_t)}.$$

We therefore obtain

$$|I_1| \lesssim \|F\|_{H^1(\Sigma_t)} \|F\|_{L^2(\Sigma_t)}.$$

In order to estimate the term I_2, we recall that $\mathrm{tr}\theta = -a\mathrm{tr}\chi + \delta^{AB} k_{AB}$. Since the bootstrap assumption (**BA2**) implies $|\mathrm{tr}\chi - 2/s| \le \mathcal{E}_0$ on each $S_{t,u'}$ and Proposition 9 implies that s, $t(p) - t$ and r are comparable, we have

$$|I_2| \lesssim (\mathcal{E}_0 \tau + 1) \int_{u_m}^{u_M} \int_{S_{t,u'}} r^{-1} |\phi F|^2 d\mu_\gamma du' + \int_{u_m}^{u_M} \int_{S_{t,u'}} |k| |\phi F|^2 d\mu_\gamma du'$$
$$\lesssim \int_{u_m}^{u_M} \int_{S_{t,u'}} r^{-1} |\phi F|^2 d\mu_\gamma du' + \|k\|_{L^3(\Sigma_t)} \|F\|^2_{L^3(\Sigma_t)}.$$

Recall that $\|k\|_{L^3(\Sigma_t)} \le C$ from Lemma 2.2 and apply Lemma 2.3 to $\|F\|^2_{L^3(\Sigma_t)}$ we obtain

$$|I_2| \lesssim \|F\|_{H^1(\Sigma_t)} \|F\|_{L^2(\Sigma_t)} + \int_{\frac{u_m+u_M}{2}}^{u_M} \int_{S_{t,u'}} r^{-1} |F|^2 d\mu_\gamma du'.$$

Now we use Lemma 7.3 again and note that Proposition 9 implies $r(t,u')^{-1} \lesssim (u'-u_m)^{-1}$, we have

$$\int_{\frac{u_m+u_M}{2}}^{u_M}\int_{S_{t,u'}} r^{-1}|F|^2 d\mu_\gamma du' \lesssim \|F\|_{H^1(\Sigma_t)}\|F\|_{L^2(\Sigma_t)}\left(\int_{\frac{u_m+u_M}{2}}^{u_M}(u'-u_m)^{-1}du'\right)^{1/2}$$
$$\lesssim \|F\|_{H^1(\Sigma_t)}\|F\|_{L^2(\Sigma_t)}.$$

Therefore

$$|I_2| \lesssim \|F\|_{H^1(\Sigma_t)}\|F\|_{L^2(\Sigma_t)}.$$

The proof of (7.5) is complete.

Applying (7.7) with $|F|$ replaced by $|F|^2$, combined with Sobolev embedding, we can obtain (7.6) in the similar fashion. □

As a consequence, we obtain

Proposition 11. *Let the bootstrap assumptions* (**BA1**)–(**BA3**) *hold on* $\mathcal{N}^-(p,\tau)$ *with* $\mathcal{E}_0\tau \le 1$. *Let* $S_t := \mathcal{N}^-(p,\tau)\cap\Sigma_t$ *and let* r *be defined by (4.2). Let* π_0 *denote the tensor* $-\nabla \log n$.

(*a*) *Let* $\underline{\pi}$ *denote either* k *,* π_0 *or* $\boldsymbol{D}_0 \log n$*, then for* $t(p)-\tau \le t \le t(p)$

$$\|\underline{\pi}\|_{L^4(S_t)} \le C, \tag{7.8}$$

$$\|r^{-1/2}\underline{\pi}\|_{L^2(S_t)} \le C. \tag{7.9}$$

(*b*) *Let* F *denotes either* $n^{-1}\nabla^2 n$ *or* $n^{-2}\nabla\dot{n}$*, then*

$$\|F\|_{L^2(\mathcal{N}^-(p,\tau))} \le C. \tag{7.10}$$

(*c*) *For* π_0*, there holds*

$$\|\nabla_L \pi_0\|_{L^2(\mathcal{N}^-(p,\tau))} + \|\boldsymbol{D}_0\pi_0\|_{L^2(\mathcal{N}^-(p,\tau))} + \|\nabla\pi_0\|_{L^2(\mathcal{N}^-(p,\tau))} \le C \tag{7.11}$$

Proof. (a) From Lemma 2.2, Proposition 3 and Lemma 3.2 it follows that $\|\underline{\pi}\|_{H^1(\Sigma_t)} \le C$. Thus (7.8) follows from (7.6) in Proposition 10 and (7.9) follows from Lemma 7.3.

(b) For $F = (n^{-1}\nabla^2 n, n^{-2}\nabla\dot{n})$ it follows from Proposition 3, Proposition 4, Lemma 3.2 and Proposition 7 that

$$\|\nabla F\|_{L_t^1 L_x^2(\mathcal{M}_*)} \le C \quad \text{and} \quad \|F\|_{L_t^\infty L_x^2(\mathcal{M}_*)} \le C.$$

Applying (7.5) to F yields

$$\|F\|^2_{L^2(\mathcal{N}^-(p,\tau))} \lesssim \|F\|_{L_t^1 H_x^1(\mathcal{M}_*)}\|F\|_{L_t^\infty L_x^2(\mathcal{M}_*)} \lesssim C.$$

(c) By straightforward calculation, symbolically we have

$$\begin{aligned}\mathbf{D}_0\pi_0 &= -n^{-2}\nabla\dot{n} + \underline{\pi}\cdot\pi_0,\\ \nabla\pi_0 &= -n^{-1}\nabla^2 n + \underline{\pi}\cdot\pi_0,\\ \nabla_L\pi_0 &= a^{-1}n^{-2}\nabla\dot{n} - a^{-1}\nabla\pi_0 - a^{-1}\underline{\pi}\cdot\pi_0.\end{aligned}$$

Therefore, (7.11) follows immediately from (7.8) and (7.10). □

8. **Estimates on the null cones**

8.1. **Structure equations on the null cones.** In Section 4 we introduced the null pair L, $\underline{L}$ on the null cone $\mathcal{N}^-(p,\tau)$ and define the null second fundamental forms χ, $\underline{\chi}$ and the Ricci coefficients ζ and $\underline{\zeta}$. For the null frame $(e_A)_{A=1,2}$, $e_3=\underline{L}$, $e_4=L$, there hold

$$\begin{aligned} \mathbf{D}_A\underline{L} &= \underline{\chi}_{AB}e_B+\zeta_A\underline{L}, & \mathbf{D}_A L &= \chi_{AB}e_B-\zeta_A L,\\ \mathbf{D}_{\underline{L}}\underline{L} &= 2\underline{\xi}_A e_A+2\omega\underline{L}, & \mathbf{D}_{\underline{L}}L &= 2\zeta_A e_A-2\omega L,\\ \mathbf{D}_L\underline{L} &= 2\underline{\zeta}_A e_A, & \mathbf{D}_L L &= 0 \end{aligned} \tag{8.1}$$

and

$$\mathbf{D}_B e_A = \nabla\!\!\!\!/_B e_A+\frac{1}{2}\chi_{AB}e_3+\frac{1}{2}\underline{\chi}_{AB}e_4, \tag{8.2}$$

$$\begin{aligned} \mathbf{D}_4 e_A &= \nabla\!\!\!\!/_4 e_A+\underline{\zeta}_A e_4,\\ \mathbf{D}_3 e_A &= \nabla\!\!\!\!/_3 e_A+\zeta_A e_3+\xi_A e_4, \end{aligned} \tag{8.3}$$

where $\nabla\!\!\!\!/$ denotes the covariant differentiation on S_t.

Let α, β, ρ and σ be the null components of $\mathbf{R}$ defined in (4.11). There hold the following structure equations on null cones (see [5, p.351–360].)

$$\frac{d\mathrm{tr}\chi}{ds}+\frac{1}{2}(\mathrm{tr}\chi)^2=-|\hat{\chi}|^2, \tag{8.4}$$

$$\frac{d\hat{\chi}_{AB}}{ds}+\mathrm{tr}\chi\hat{\chi}_{AB}=\alpha_{AB}, \tag{8.5}$$

$$\frac{d}{ds}\zeta_A=-\chi_{AB}\zeta_B+\chi_{AB}\underline{\zeta}_B-\beta_A, \tag{8.6}$$

$$\frac{d}{ds}\mathrm{tr}\underline{\chi}+\frac{1}{2}\mathrm{tr}\chi\mathrm{tr}\underline{\chi}=2\,\mathrm{div}\,\underline{\zeta}-\hat{\chi}\cdot\underline{\hat{\chi}}+2|\underline{\zeta}|^2+2\rho. \tag{8.7}$$

Moreover, ζ verifies the following Hodge system

$$\mathrm{div}\,\zeta=-\mu-\rho+\frac{1}{2}\hat{\chi}\cdot\underline{\hat{\chi}}-|\zeta|^2-\frac{1}{2}a\delta\mathrm{tr}\chi-a\lambda\mathrm{tr}\chi, \tag{8.8}$$

$$\mathrm{curl}\,\zeta=\sigma-\frac{1}{2}\hat{\chi}\wedge\underline{\hat{\chi}}, \tag{8.9}$$

where μ and $\underline{\mu}$ are the mass aspect functions defined by

$$\mu=-\frac{1}{2}\mathbf{D}_3\mathrm{tr}\chi+\frac{a^2}{4}(\mathrm{tr}\chi)^2-\omega\mathrm{tr}\chi, \tag{8.10}$$

$$\underline{\mu}=\mathbf{D}_4\mathrm{tr}\underline{\chi}+\frac{1}{2}\mathrm{tr}\chi\cdot\mathrm{tr}\underline{\chi}, \tag{8.11}$$

$$\omega=\frac{1}{2}(\mathbf{D}_3\log a+ak_{NN}-a\pi_{0N}). \tag{8.12}$$

Let N be the unit inward normal to S_t in Σ_t and let θ be the second fundamental form of S_t, i.e. $\theta_{AB}=g(\nabla_A N,e_B)$. Then there hold

$$\nabla_N e_A=\nabla\!\!\!\!/_N e_A+a^{-1}\nabla\!\!\!\!/_A aN, \tag{8.13}$$

$$\nabla_A N=\theta_{AB}e_B, \tag{8.14}$$

$$\nabla_B e_A=\nabla\!\!\!\!/_B e_A-\theta_{AB}N, \tag{8.15}$$

$$\nabla_N N=-a^{-1}\nabla\!\!\!\!/_A a e_A. \tag{8.16}$$

We introduce the new null pair $L' := \mathbf{T} + N$, $\underline{L}' := \mathbf{T} - N$. Then $L = -a^{-1}L'$ and $\underline{L} = -a\underline{L}'$. Let $\chi', \underline{\chi}', \zeta', \underline{\zeta}', \nu$ be the Ricci coefficients corresponding to the null frame $(e_A)_{A=1,2}$, $e_3' = \underline{L}', e_4' = L'$. Then

$$\chi = -a^{-1}\chi', \qquad \underline{\chi} = -a\underline{\chi}', \qquad \zeta = \zeta', \qquad \underline{\zeta} = \underline{\zeta}'$$

and

$$\chi'_{AB} = \theta_{AB} - k_{AB}, \tag{8.17}$$

$$\underline{\chi}'_{AB} = -\theta_{AB} - k_{AB}, \tag{8.18}$$

$$\zeta'_A = \nabla\!\!\!/_A \log a + \epsilon_A, \tag{8.19}$$

$$\underline{\zeta}'_A = \nabla\!\!\!/_A \log n - \epsilon_A, \tag{8.20}$$

$$\nu = -\nabla\!\!\!/_N \log n + \delta - \lambda. \tag{8.21}$$

8.2. **Proof of Theorem 4.5.** The main purpose of this subsection is to prove Theorem 4.5 concerning the boundedness of $\mathcal{N}_1[\pi\!\!\!/]$ under the bootstrap assumptions (**BA1**)–(**BA3**) on $\mathcal{N}^-(p,\tau)$ with $0 < \tau \le i_*$ and $\mathcal{E}_0\tau \le 1$ for any $p \in \mathcal{M}_I$, where $\pi\!\!\!/$ is defined by (4.14) and the Sobolev norm $\mathcal{N}_1[F]$ for any S_t tangent tensor field F is defined by (4.12). We can restate Theorem 4.5 in the following form, since the estimates for λ are trivial.

Proposition 12. *Let $\pi\!\!\!/$ be the S_t tangent tensor field defined in (4.14), and let $\bar\pi := (k, -\nabla \log n)$. Then, under the bootstrap assumptions* (**BA1**)–(**BA4**) *with $\mathcal{E}_0\tau \le 1$, there hold*

$$\|r^{-1}\bar\pi\|_{L^2(\mathcal{N}^-(p,\tau))} \le C, \tag{8.22}$$

$$\|\nabla\!\!\!/\pi\!\!\!/\|_{L^2(\mathcal{N}^-(p,\tau))} \le C, \tag{8.23}$$

$$\|\nabla\!\!\!/_L\pi\!\!\!/\|_{L^2(\mathcal{N}^-(p,\tau))} \le C. \tag{8.24}$$

We have obtained in Theorem 4.4 and (7.11) that

$$\|\nabla\!\!\!/\bar\pi\|_{L^2(C_u)} + \|\nabla_L\bar\pi\|_{L^2(C_u)} \le C. \tag{8.25}$$

In view of (8.14), (8.15) and (8.1), (8.3), we can symbolically write

$$\nabla\!\!\!/\pi\!\!\!/ = \nabla\!\!\!/\bar\pi + \mathrm{tr}\theta \cdot \pi\!\!\!/ + \hat\theta \cdot \pi\!\!\!/ \tag{8.26}$$

and also in view of $\frac{dt}{ds} = -(an)^{-1}$,

$$\nabla\!\!\!/_L\pi\!\!\!/ = \nabla_L\bar\pi + \pi\!\!\!/ \cdot \underline{\zeta} + (an)^{-1}. \tag{8.27}$$

In order to show Proposition 12, we need three auxiliary lemmas. We will use the following norms for Σ_t tangent tensor fields F on null cones $\mathcal{N}^-(p,\tau)$

$$\|F\|^q_{L^q_x L^\infty_t(\mathcal{N}^-(p,\tau))} := \int_{\mathbb{S}^2} \sup_{t\in\Gamma_\omega} \left(v_t|F|^q_g\right) d\mu_{\mathbb{S}^2},$$

$$\|F\|^q_{L^q_\omega L^\infty_t(\mathcal{N}^-(p,\tau))} := \int_{\mathbb{S}^2} \sup_{t\in\Gamma_\omega} |F|^q_g d\mu_{\mathbb{S}^2}.$$

where v_t is defined by (7.1), and Γ_ω, $\omega \in \mathbb{S}^2$, denotes the portion of an incoming null geodesic initiating from p in the time slab $[t(p) - \tau, t(p)]$. In the following argument we will suppress $\mathcal{N}^-(p,\tau)$ in these norms for simplicity.

Lemma 8.1. *For any S_t tangent tensor field F, there hold the estimates*

$$\|r^{-1/2}F\|_{L_x^2 L_t^\infty} + \|F\|_{L_x^4 L_t^\infty} \lesssim \mathcal{N}_1[F], \tag{8.28}$$

$$\|F\|^2_{L_x^4 L_t^\infty} \lesssim \left(\|\nabla\!\!\!/_L F\|_{L^2} + \|r^{-1}F\|_{L^2}\right)\|F\|_{L_\omega^\infty L_t^2}. \tag{8.29}$$

Proof. We refer to [7, 13] for the proof of (8.28). In the following we will prove (8.29). Let v_t be defined by (7.1). We first integrate along any past null geodesic initiating from p to get

$$v_t|F|^4 = \lim_{t\to t(p)}(v_t|F|^4) - \int_t^{t(p)} \frac{d}{dt'}(v_{t'}|F|^4)dt'. \tag{8.30}$$

For the estimate of the first term on the right of (8.30), we proceed as follows. Let φ be a smooth cut-off function defined on $[t(p)-\tau, t(p)]$ verifying $0\le\varphi\le 1$, $\varphi(t(p))=1$ and $\mathrm{supp}\varphi \subset [t(p)-\tau/2, t(p)]$. Then

$$\lim_{t\to t(p)} v_t|F|^4 = \int_{t(p)-\tau}^{t(p)} \left(\frac{d}{dt}(v_t|F|^4)\varphi^4 + 4v_t|F|^4\varphi^3\frac{d}{dt}\varphi\right)dt. \tag{8.31}$$

Since $|\frac{d}{dt}\varphi| \lesssim (t(p)-t)^{-1}$, we have from Lemma 7.1 that $|\frac{d}{dt}\varphi|v_t^{\frac{1}{2}} \lesssim 1$. Using $0\le\varphi\le 1$, it then follows from (8.30) and (8.31) that

$$\|F\|^4_{L_x^4 L_t^\infty} = \int_{\mathbb{S}^2} \sup_{t(p)-\tau\le t\le t(p)} (v_t|F|^4) \lesssim I + II, \tag{8.32}$$

where

$$I = \int_{\mathbb{S}^2}\int_{t(p)-\tau}^{t(p)} \left|\frac{d}{dt}(v_t|F|^4)\right| dt, \qquad II = \int_{\mathbb{S}^2}\int_{t(p)-\tau}^{t(p)} v_t^{1/2}|F|^4.$$

Since

$$\frac{d}{dt}(v_t|F|^4) = -na\left(\mathrm{tr}\chi v_t|F|^4 + 4v_t|F|^2\nabla\!\!\!/_L F\cdot F\right),$$

we have

$$\begin{aligned} I &\lesssim \left(\|v_t^{1/2}\nabla\!\!\!/_L F\|_{L_\omega^2 L_t^2} + \|\mathrm{tr}\chi v_t^{1/2}F\|_{L_\omega^2 L_t^2}\right)\|F\|_{L_\omega^\infty L_t^2}\|v_t^{1/2}|F|^2\|_{L_\omega^2 L_t^\infty} \\ &\lesssim \left(\|\nabla\!\!\!/_L F\|_{L^2} + \|\mathrm{tr}\chi F\|_{L^2}\right)\|F\|_{L_\omega^\infty L_t^2}\|F\|^2_{L_x^4 L_t^\infty}. \end{aligned}$$

By the bootstrap assumption (**BA2**) and Lemma 7.1 we have

$$\begin{aligned} \|\mathrm{tr}\chi F\|_{L^2} &\lesssim \left\|\mathrm{tr}\chi - \frac{2}{s}\right\|_{L^\infty}\tau\|r^{-1}F\|_{L^2} + \|r^{-1}F\|_{L^2} \\ &\lesssim (\mathcal{E}_0\tau+1)\|r^{-1}F\|_{L^2} \lesssim \|r^{-1}F\|_{L^2}. \end{aligned}$$

Therefore

$$I \lesssim \left(\|\nabla\!\!\!/_L F\|_{L^2} + \|r^{-1}F\|_{L^2}\right)\|F\|_{L_\omega^\infty L_t^2}\|F\|^2_{L_x^4 L_t^\infty}.$$

It is easy to see that

$$|II| \lesssim \|F\|_{L_\omega^2 L_t^2}\|F\|_{L_\omega^\infty L_t^2}\|v_t^{1/2}|F|^2\|_{L_\omega^2 L_t^\infty} \lesssim \|r^{-1}F\|_{L^2}\|F\|_{L_\omega^\infty L_t^2}\|F\|^2_{L_x^4 L_t^\infty}.$$

Combining the estimates for I and II with (8.32) gives (8.29). $\square$

Lemma 8.2. *For any S_t tangent tensor field F verifying*

$$\nabla\!\!\!/_L F + \frac{m}{2} tr\chi F = G \cdot F + H \tag{8.33}$$

with $m \ge 1$ an integer and G a tensor field of suitable type, if $\lim_{t \to t(p)} r(t)^m F = 0$ and $\sup_{\omega \in \mathbb{S}^2} \int_{t(p)-\tau}^{t(p)} na|G|^2 dt \le \Delta_0^2$, the following estimates hold

$$\|F\|_{L^2_\omega L^2_t} \lesssim e^{C\Delta_0 \tau^{1/2}} \|H\|_{L^2}, \tag{8.34}$$

$$\|r^{\frac{1}{2}} F\|_{L^2_\omega L^\infty_t} \lesssim e^{C\Delta_0 \tau^{1/2}} \|H\|_{L^2}. \tag{8.35}$$

Proof. In what follows, we will use Lemma 7.1 to compare $v_t^{1/2}$, r, s and $t(p) - t$ if necessary. Since $\frac{d}{dt} v_t = -na\mathrm{tr}\chi v_t$, along any past null geodesic initiating from p we have

$$\frac{d}{dt}(v_t^m |F|^2) = -2nav_t^m \langle H + F \cdot G, F \rangle$$

With the help of the $\lim_{t \to t(p)} r^m |F| = 0$, it follows for $t(p) - \tau \le t \le t(p)$ that

$$v_t^m |F|^2 = 2 \int_t^{t(p)} nav_{t'}^m \langle H + F \cdot G, F \rangle \le 2 \int_t^{t(p)} nav_{t'}^m \left(|F||H| + |F|^2 |G| \right).$$

By a simple argument we can derive

$$v_t^{m/2} |F| \le \exp\left(\int_t^{t(p)} |G| na \right) \int_t^{t(p)} nav_{t'}^{m/2} |H| \exp\left(- \int_{t'}^{t(p)} na|G| \right) dt'.$$

In view of $\sup_{\omega \in \mathbb{S}^2} \int_{t(p)-\tau}^{t(p)} na|G|^2 dt \le \Delta_0^2$, we have $\exp(\int_t^{t(p)} na|G|) \le e^{C\Delta_0 \tau^{1/2}}$. Thus by using Lemma 7.1 and $m \ge 1$, we have

$$\begin{aligned} |F| &\le e^{C\Delta_0 \tau^{1/2}} v_t^{-m/2} \int_t^{t(p)} v_{t'}^{m/2} |H| na dt' \\ &\lesssim e^{C\Delta_0 \tau^{1/2}} (t(p) - t)^{-1} \int_t^{t(p)} r|H| dt'. \end{aligned} \tag{8.36}$$

To derive (8.34), we integrate the above inequality along a null geodesic initiating from vertex p. By the Hardy-Littlewood inequality

$$\left\| \frac{1}{s} \int_0^s |f| \right\|_{L^2_s} \lesssim \|f\|_{L^2_s}$$

it follows that

$$\begin{aligned} \|F\|_{L^2_t} &\lesssim e^{C\Delta_0 \tau^{1/2}} \left\| \frac{1}{t(p) - t} \int_t^{t(p)} r|H| \right\|_{L^2_t} \\ &\lesssim e^{C\Delta_0 \tau^{1/2}} \|rH\|_{L^2_t}. \end{aligned} \tag{8.37}$$

Integrating (8.37) with respect to the angular variable $\omega \in \mathbb{S}^2$ yields (8.34).

Next we multiply (8.36) by $r^{\frac{1}{2}}$ to obtain

$$\sup_{t(p)-\tau \le t \le t(p)} r^{\frac{1}{2}} |F| \lesssim e^{C\Delta_0 \tau^{1/2}} \|rH\|_{L^2_t},$$

which, by taking the L^2_ω norm, gives (8.35). $\square$

In view of (8.5) and Lemma 8.2, we are able to prove the following estimates for $\hat{\chi}$.

Lemma 8.3. *For $\hat{\chi}$ there hold the estimates*

$$\|r^{-1}\hat{\chi}\|_{L^2} + \|r^{1/2}\hat{\chi}\|_{L^2_\omega L^\infty_t} + \|\nabla\!\!\!/_L\hat{\chi}\|_{L^2} \le C, \tag{8.38}$$

$$\|\hat{\chi}\|_{L^4_x L^\infty_t} \le C\mathcal{E}_0^{1/4}. \tag{8.39}$$

Proof. We will use the transport equation (8.5), i.e.

$$\nabla\!\!\!/_L\hat{\chi} + \mathrm{tr}\chi\hat{\chi} = \alpha. \tag{8.40}$$

Recall that $r\hat{\chi} \to 0$ as $t \to t(p)$, see [13]. Recall also that $\|\alpha\|_{L^2} \le C$, see Theorem 4.4. It then follows from Lemma 8.2 that

$$\|r^{1/2}\hat{\chi}\|_{L^2_\omega L^\infty_t} + \|\hat{\chi}\|_{L^2_\omega L^2_t} \le C.$$

Next we use (8.40) again to estimate $\|\nabla\!\!\!/_L\hat{\chi}\|_{L^2}$. With the help of the bootstrap assumption (**BA2**) and the comparability of r, s and $t(p)-t$ given in Lemma 7.1, we have

$$\|\mathrm{tr}\chi\,\hat{\chi}\|_{L^2} \lesssim \left\|\mathrm{tr}\chi - \frac{2}{s}\right\|_{L^\infty} \|r\hat{\chi}\|_{L^2_t L^2_\omega} + \|r^{-1}\hat{\chi}\|_{L^2} \le C.$$

Thus, from (8.40) it follows

$$\|\nabla\!\!\!/_L\hat{\chi}\|_{L^2} \lesssim \|\mathrm{tr}\chi\hat{\chi}\|_{L^2} + \|\alpha\|_{L^2} \le C.$$

We therefore complete the proof of (8.38).

By making use of (8.29) and (8.38) together with the bootstrap assumption (**BA3**) we obtain

$$\|\hat{\chi}\|_{L^4_x L^\infty_t} \lesssim (\|\nabla\!\!\!/_L\hat{\chi}\|_{L^2} + \|r^{-1}\hat{\chi}\|_{L^2})^{\frac{1}{2}} \|\hat{\chi}\|^{\frac{1}{2}}_{L^\infty_\omega L^2_t} \le C\mathcal{E}_0^{1/4}$$

which gives (8.39). $\square$

Now we are ready to complete the proof of Proposition 12.

Proof of Proposition 12. We first prove (8.22). Let $|\bar{\pi}| := |\bar{\pi}|_g$. It is easy to check

$$\nabla\!\!\!/_L(s^{-1}|\bar{\pi}|^2) + \mathrm{tr}\chi s^{-1}|\bar{\pi}|^2 = s^{-1}(\mathrm{tr}\chi - \frac{2}{s})|\bar{\pi}|^2 + s^{-2}|\bar{\pi}|^2_g + 2s^{-1}\nabla_L\bar{\pi}\cdot\bar{\pi}.$$

We integrate the above equation along the null cone $\mathcal{N}^-(p,\tau)$. By Lemma 7.1, it is easy to see $\int_{S_t} s^{-1}|\bar{\pi}|^2 \to 0$ as $t \to t(p)$. Therefore, by integration by parts we obtain

$$\int_{\mathcal{N}^-(p,\tau)} \left(s^{-2}|\bar{\pi}|^2 + s^{-1}(\mathrm{tr}\chi - \frac{2}{s})|\bar{\pi}|^2 + 2s^{-1}\nabla_L\bar{\pi}\cdot\bar{\pi}\right) nad\mu_\gamma dt = \int_{S_{t(p)-\tau}} s^{-1}|\bar{\pi}|^2.$$

By Lemma 7.1 and (7.9) in Proposition 11 we have

$$\left|\int_{S_{t(p)-\tau}} s^{-1}|\bar{\pi}|^2\right| \lesssim \|r^{-1/2}\bar{\pi}\|^2_{L^2(S_{t(p)-\tau})} \le C.$$

By (**BA2**), Lemma 7.1 and (7.9),

$$\left|\int_{\mathcal{N}^-(p,\tau)} nas^{-1}(\mathrm{tr}\chi - \frac{2}{s})|\bar{\pi}|^2 d\mu_\gamma dt\right| \le C\mathcal{E}_0\tau \le C.$$

By (8.25) we have

$$\left|\int_{\mathcal{N}^-(p,\tau)} s^{-1}\nabla_L \bar{\pi}\cdot\bar{\pi} n a d\mu_\gamma dt\right| \lesssim \|\nabla_L\bar{\pi}\|_{L^2}\|s^{-1}\bar{\pi}\|_{L^2} \le C\|s^{-1}\bar{\pi}\|_{L^2}.$$

Therefore

$$\|s^{-1}\bar{\pi}\|_{L^2}^2 \le C + C\|s^{-1}\bar{\pi}\|_{L^2} \tag{8.41}$$

which implies $\|s^{-1}\bar{\pi}\|_{L^2}\le C$. Consequently, in view of Lemma 7.1, (8.22) follows. As a byproduct, we have from (**BA2**) and Lemma 7.1 that

$$\|\mathrm{tr}\chi\bar{\pi}\|_{L^2} \lesssim \|s^{-1}\bar{\pi}\|_{L^2} + \left\|\mathrm{tr}\chi - \frac{2}{s}\right\|_{L^\infty}\tau\|s^{-1}\bar{\pi}\|_{L^2} \le C(1+\mathcal{E}_0\tau)\le C. \tag{8.42}$$

Next we will show (8.23). we will use the equation (8.26), i.e.

$$\nabla\!\!\!/ \pi\!\!\!/ = \nabla\!\!\!/ \bar{\pi} + \mathrm{tr}\theta\cdot \pi\!\!\!/ + \hat{\theta}\cdot\pi\!\!\!/. \tag{8.43}$$

Using $\theta_{AB} = -a\chi_{AB}+k_{AB}$, we have from (7.8) and (8.39) that

$$\|\hat{\theta}\cdot\pi\!\!\!/\|_{L^2}\lesssim \|\pi\!\!\!/\|_{L^4}\left(\|k\|_{L^4}+\|\hat{\chi}\|_{L^4}\right)\le C\left(\mathcal{E}_0^{1/4}+1\right)\tau^{1/2}\le C.$$

Since $\mathrm{tr}\theta = -a\mathrm{tr}\chi + \delta^{AB}k_{AB}$, we have from (7.8) and (8.42) that

$$\|\mathrm{tr}\theta\pi\!\!\!/\|_{L_t^2L_x^2}\lesssim \|k\|_{L^4}\|\pi\!\!\!/\|_{L^4}+\|\mathrm{tr}\chi\pi\!\!\!/\|_{L^2}\le C.$$

Consequently, in view of (8.25) and (8.43), (8.23) follows immediately.

In view of (8.27) and (8.20), (8.24) follows immediately from (8.25) and (7.8). □

8.3. **Estimates for Ricci coeffients.**

Lemma 8.4. *For the Ricci coefficient ζ and the null lapse a there hold*

$$\|r^{\frac12}\zeta\|_{L_\omega^2L_t^\infty}+\|r^{-1}\zeta\|_{L^2}+\|\nabla\!\!\!/_L\zeta\|_{L^2}\le C, \tag{8.44}$$

$$\|r^{\frac12}\nabla\!\!\!/\log a\|_{L_\omega^2L_t^\infty}+\|r^{-1}\nabla\!\!\!/\log a\|_{L^2}+\|\nabla\!\!\!/_L\nabla\!\!\!/\log a\|_{L^2}\le C. \tag{8.45}$$

Proof. From the transport equation (8.6) we have

$$\nabla\!\!\!/_L\zeta + \frac12\mathrm{tr}\chi\cdot\zeta = -\hat{\chi}\cdot\zeta + \chi\cdot\underline{\zeta} - \beta. \tag{8.46}$$

Since (**BA3**) implies $\|\hat{\chi}\|_{L_\omega^\infty L_t^2}\le\mathcal{E}_0^{1/2}$ with $\mathcal{E}_0\tau\le 1$, it follows from Lemma 8.2 and the relation $\chi=\hat{\chi}+\frac12\mathrm{tr}\chi\,\gamma$ that

$$\|r^{\frac12}\zeta\|_{L_\omega^2L_t^\infty}+\|r^{-1}\zeta\|_{L^2}\lesssim\|\beta\|_{L^2}+\|\hat{\chi}\cdot\underline{\zeta}\|_{L^2}+\|\mathrm{tr}\chi\cdot\underline{\zeta}\|_{L^2}$$

From Theorem 4.4 we have $\|\beta\|_{L^2}\le C$. Recall that $\underline{\zeta}=\nabla\!\!\!/\log n-\epsilon$ which is a combination of terms in $\pi\!\!\!/$. By (8.42) we have $\|\mathrm{tr}\chi\,\underline{\zeta}\|_{L^2}\le C$. Therefore

$$\|r^{1/2}\zeta\|_{L_\omega^2L_t^\infty}+\|r^{-1}\zeta\|_{L^2}\le C\left(\mathcal{E}_0\tau+1\right)+\|\hat{\chi}\cdot\underline{\zeta}\|_{L^2}.$$

In view of (7.8) in Proposition 11, (8.39) in Lemma 8.3, and $\mathcal{E}_0\tau\le1$, we have

$$\|r^{\frac12}\zeta\|_{L_\omega^2L_t^\infty}+\|r^{-1}\zeta\|_{L^2}\le C+\tau^{1/2}\|\hat{\chi}\|_{L_x^4L_t^\infty}\|\underline{\zeta}\|_{L_t^\infty L_x^4}\le C.$$

Consequently, it follows from (8.46), (**BA2**) and (**BA3**) that $\|\nabla\!\!\!/_L\zeta\|_{L^2}\le C$. We thus obtain (8.44).

In order to show (8.45), we use the relation $\zeta=\nabla\!\!\!/\log a+\epsilon$. By Proposition 12,

$$\|r^{\frac12}\epsilon\|_{L_\omega^2L_t^\infty}+\|\epsilon\|_{L_\omega^2L_t^2}+\|\nabla\!\!\!/_L\epsilon\|_{L^2}\le C.$$

Thus, the estimates for $\nabla\!\!\!/ \log a$ follows. □

Lemma 8.5. *For the $\underline{\mu}$ defined by (8.11) there holds $\|\underline{\mu}\|_{L^2} \le C$ on $\mathcal{N}^-(p,\tau)$.*

Proof. Recall that by (8.7), $\underline{\mu} = 2\text{div}\underline{\zeta} - \hat{\chi}\cdot\underline{\hat{\chi}} + 2|\underline{\zeta}|^2 + 2\rho$. We have from Theorem 4.4, Proposition 11 and Theorem 4.5 that

$$\|\underline{\mu}\|_{L^2} \lesssim \|\nabla\!\!\!/ \underline{\zeta}\|_{L^2} + \|\underline{\zeta}\|_{L^4}^2 + \|\rho\|_{L^2} + \|\hat{\chi}\cdot\underline{\hat{\chi}}\|_{L^2} \lesssim C + \|\hat{\chi}\cdot\underline{\hat{\chi}}\|_{L^2}.$$

Recall also the relation $\underline{\hat{\chi}}\,' = -\hat{\chi}' - 2\hat{\eta}$, we have from (8.39) and Proposition 11 that

$$\|\underline{\mu}\|_{L^2} \lesssim C + \|\hat{\chi}\|_{L^4}\left(\|\hat{\chi}\|_{L^4} + \|k\|_{L^4}\right) \le C.$$

The proof is thus complete. □

In the following we summarize the estimates obtained so far in this section.

Proposition 13. *There exists universal constants $\delta_0 > 0$ and $C_* > 0$ such that, under the bootstrap assumptions* (**BA1**)–(**BA3**) *with $\mathcal{E}_0\tau \le 1$, if $\tau < \min\{i_*, \delta_0\}$ then there hold*

$$\|r^{-\frac{1}{2}}\underline{\pi}\|_{L^2(S_{t,u})} \le C, \tag{8.47}$$

$$\|\underline{\pi}\|_{L^4(S_{t,u})} \le C, \tag{8.48}$$

$$\mathcal{N}_1[\pi\!\!\!/](p,\tau) \le C, \tag{8.49}$$

$$\|n^{-1}\nabla^2 n, n^{-2}\nabla\dot{n}\|_{L^2} \le C, \tag{8.50}$$

$$\|r^{\frac{1}{2}}(\hat{\chi}, \bar{\pi}, \zeta, \nabla\!\!\!/ \log a, \hat{\theta})\|_{L^2_\omega L^\infty_t} \le C, \tag{8.51}$$

$$\|(\hat{\chi}, \bar{\pi}, \zeta, \nabla\!\!\!/ \log a, \hat{\theta})\|_{L^2_t L^2_\omega} \le C, \tag{8.52}$$

$$\|\nabla\!\!\!/_L(\hat{\chi}, \zeta, \nabla\!\!\!/ \log a, \hat{\theta})\|_{L^2} \le C, \tag{8.53}$$

where $\underline{\pi} = (n^{-1}\partial_t \log n, \bar{\pi})$.

The above estimates provide the intermediate steps toward the proof of Theorem 4.6. The complete proof however requires more estimates on $\hat{\chi}$, ζ and $\underline{\zeta}$ as follows. Since the arguments are rather lengthy, we will report them in [14].

Proposition 14. *There exists universal constants $\delta_0 > 0$ and $C_* > 0$ such that, under the bootstrap assumptions* (**BA1**)–(**BA4**) *with $\mathcal{E}_0\tau \le 1$, if $\tau < \min\{i_*, \delta_0\}$ then there hold*

$$\left\| tr\chi - \frac{2}{s} \right\|_{L^\infty} \le C_*, \tag{8.54}$$

$$\|\hat{\chi}\|_{L^\infty_\omega L^2_t} + \|\zeta\|_{L^\infty_\omega L^2_t} \le C_*, \tag{8.55}$$

$$\|\nu\|_{L^\infty_\omega L^2_t} + \|\underline{\zeta}\|_{L^\infty_\omega L^2_t} \le C_*, \tag{8.56}$$

$$\mathcal{N}_1[\hat{\chi}, \zeta, \nabla\!\!\!/ \log a, \hat{\theta}](p,\tau) \le C_*, \tag{8.57}$$

$$\|r^{\frac{1}{2}}(\nabla\!\!\!/ tr\chi, \mu)\|_{L^2_x L^\infty_t} + \|(\nabla\!\!\!/ tr\chi, \mu)\|_{L^2} \le C_*, \tag{8.58}$$

on the null cone $\mathcal{N}^-(p,\tau)$ for all $p \in \mathcal{M}_I$.

The estimates in Proposition 13 and Proposition 14 gives Theorem 4.6. Thus, we may use a bootstrap argument, as explained in Section 4, to conclude that all the estimates in the above two propositions hold on the null cones $\mathcal{N}^-(p,\tau)$ for all $p \in \mathcal{M}_I$ with $\tau = \min\{i_*, \delta_*\}$ for some universal constant $\delta_* > 0$.

We conclude this section with an application to estimate $\|\underline{\pi}\|_{L_u^2 L_\omega^2(\mathrm{Int}(S_{t,u}))}$, where, for any Σ tangent tensor F,

$$\|F\|^2_{L_u^2 L_\omega^2(\mathrm{Int} S_{t,u})} = \int_{u_m}^u \int_{S_{t,u'}} r'^{-2} |F|_g^2 a d\mu_\gamma du'$$

with $r' = r(t, u')$.

Proposition 15. *For $\underline{\pi} = (n^{-1}\partial_t \log n, \bar{\pi})$, there holds*

$$\|\underline{\pi}\|_{L_u^2 L_\omega^2(Int(S_{t,u}))} \le C. \tag{8.59}$$

Proof. From (8.4), (8.10), (8.12) and (8.8), we can derive

$$\nabla\!\!\!\!/_N \mathrm{tr}\chi' + \frac{1}{2}(\mathrm{tr}\chi')^2 = -\frac{1}{2}\delta \mathrm{tr}\chi' + 2\lambda \mathrm{tr}\chi' - \hat{\chi}'(\hat{\chi}' + \hat{\eta}) - (\,\mathrm{div}\,\zeta + |\zeta|^2 + \rho), \tag{8.60}$$

which, multiplied by $|\underline{\pi}| := |\underline{\pi}|_g$, implies

$$\begin{aligned}
&\nabla_N(\mathrm{tr}\chi' |\underline{\pi}|_g{}^2) + \mathrm{tr}\theta(\mathrm{tr}\chi'|\underline{\pi}|_g^2) - \frac{1}{2}|\mathrm{tr}\chi'\underline{\pi}|_g^2 \\
&\quad = \left\{ -\frac{3}{2}\delta \mathrm{tr}\chi' - \hat{\chi}'(\hat{\chi}' + \hat{\eta}) - (\,\mathrm{div}\,\zeta + |\zeta|^2 + \rho) \right\} |\underline{\pi}|^2 + 2\mathrm{tr}\chi' \nabla_N \underline{\pi} \cdot \underline{\pi},
\end{aligned}$$

In view of Lemma 7.2, integrating the above equation over $\mathrm{Int}(S_{t,u})$ gives

$$\begin{aligned}
&\frac{1}{2}\int_{u_m}^u \int_{S_{t,u'}} (\mathrm{tr}\chi')^2 |\underline{\pi}|^2 a d\mu_\gamma du' \\
&\qquad = -\int_{S_{t,u}} \mathrm{tr}\chi'|\underline{\pi}|^2 + \int_{u_m}^u \int_{S_{t,u'}} \left(-2\nabla_N \underline{\pi} \cdot \mathrm{tr}\chi'\underline{\pi} + \rho|\underline{\pi}|^2\right) a d\mu_\gamma du' \\
&\qquad + \int_{u_m}^u \int_{S_{t,u'}} \left(\frac{3}{2}\delta \mathrm{tr}\chi' + |\zeta|^2 + \hat{\chi}'(\hat{\chi}' + \hat{\eta})\right) |\underline{\pi}|^2 a d\mu_\gamma du' \\
&\qquad + \int_{u_m}^u \int_{S_{t,u'}} -\zeta \cdot \nabla\!\!\!\!/ (|\underline{\pi}|^2 a) d\mu_\gamma du'
\end{aligned} \tag{8.61}$$

By (**BA2**), Lemma 7.1 and (7.9),

$$\left| \int_{S_{t,u}} \mathrm{tr}\chi'|\underline{\pi}|^2 d\mu_\gamma \right| \lesssim \|r^{-1/2}\underline{\pi}\|^2_{L^2(S_{t,u})} \le C.$$

By Lemma 2.2, Proposition 3 and (3.16),

$$\left| \int_{u_m}^u \int_{S_{t,u'}} \nabla_N \underline{\pi} \cdot \mathrm{tr}\chi' \underline{\pi} a d\mu_\gamma du' \right| \lesssim \|\nabla_N \underline{\pi}\|_{L^2(\Sigma_t)} \|\mathrm{tr}\chi'\underline{\pi}\|_{L^2(\Sigma_t)} \le C\|\mathrm{tr}\chi'\underline{\pi}\|_{L^2(\Sigma_t)}$$

and

$$\begin{aligned}
\left| \int_{u_m}^{u'} \int_{S_{t,u}} \frac{3}{2}\delta \mathrm{tr}\chi'|\underline{\pi}|^2 a d\mu_\gamma du' \right| &\lesssim \|k\|_{L^6(\Sigma_t)} \|\underline{\pi}\|^2_{L^6(\Sigma_t)} + \|\underline{\pi}\|^3_{L^3(\mathrm{Int}(S_{t,u}))} \\
&\lesssim (\|\nabla k\|_{L^2(\Sigma_t)} + \|\underline{\pi}\|_{H^1(\Sigma_t)}) \|\underline{\pi}\|^2_{H^1(\Sigma_t)} \\
&\le C.
\end{aligned}$$

By Lemma 2.1 and (7.8),

$$\left|\int_{u_m}^u \int_{S_{t,u'}} \rho|\underline{\pi}|^2 a d\mu_\gamma du'\right| \lesssim \|\rho\|_{L^2(\Sigma_t)} \|\underline{\pi}\|^2_{L^4(\mathrm{Int} S_{t,u})} \le C(u-u_m)^{1/2}.$$

By (8.19) we have

$$\begin{aligned}\left|\int_{u_m}^u \int_{S_{t,u'}} \zeta \nabla\!\!\!\!/ (a|\underline{\pi}|^2) d\mu_\gamma du'\right| &= \left|\int_{u_m}^u \int_{S_{t,u'}} (\nabla\!\!\!\!/ \log a |\underline{\pi}|^2 \zeta + \nabla\!\!\!\!/ |\underline{\pi}|^2 \zeta) a d\mu_\gamma du'\right| \\ &\lesssim \|\nabla\!\!\!\!/ \underline{\pi}\|_{L^2(\mathrm{Int}(S_{t,u}))} \sup_{u_m\le u'\le u} \left(\|\underline{\pi}\|_{L^4(S_{t,u'})} \|\zeta\|_{L^4(S_{t,u'})}\right) (u-u_m)^{1/2} \\ &\quad + \int_{u_m}^u \int_{S_{t,u'}} (|\zeta|^2 |\underline{\pi}|^2 + |\zeta||\underline{\pi}|^3) a d\mu_\gamma du'.\end{aligned}$$

In view of Lemma 2.2, Proposition 3 and (3.16) we derive

$$\|\nabla\!\!\!\!/ \underline{\pi}\|_{L^2(\mathrm{Int}(S_{t,u}))} \le \|\nabla \underline{\pi}\|_{L^2(\Sigma_t)} \le C,$$

while in view of (8.57), (8.28) and (7.8) we have

$$\sup_{u_m\le u'\le u} \|\zeta\|_{L^4(S_{t,u'})} \le C, \qquad \sup_{u_m\le u'\le u} \|\underline{\pi}\|_{L^4(S_{t,u'})} \le C.$$

Consequently,

$$\begin{aligned}\int_{u_m}^u \int_{S_{t,u'}} \left(|\zeta|^2 |\underline{\pi}|^2 + |\zeta||\underline{\pi}|^3\right) a d\mu_\gamma du' &\lesssim \sup_{u_m\le u'\le u} \left(\|\zeta\|^2_{L^4(S_{t,u'})} \|\underline{\pi}\|^2_{L^4(S_{t,u'})}\right)(u-u_m) \\ &\quad + \sup_{u_m\le u'\le u} \left(\|\zeta\|_{L^4(S_{t,u'})} \|\underline{\pi}\|^3_{L^4(S_{t,u'})}\right)(u-u_m) \\ &\le C(u-u_m).\end{aligned}$$

Therefore, we obtain

$$\left|\int_{u_m}^u \int_{S_{t,u'}} \zeta \nabla\!\!\!\!/ (a|\underline{\pi}|_g^2) d\mu_\gamma du'\right| \le C(1+(u-u_m)^{1/2})(u-u_m)^{1/2}.$$

In view of (8.57), (8.28) and (7.8), by a similar argument we obtain

$$\begin{aligned}&\left|\int_{u_m}^u \int_{S_{t,u'}} (|\zeta|^2 + \hat{\chi}'(\hat{\chi}' + \hat{\eta}))|\underline{\pi}|^2 a d\mu_\gamma du'\right| \\ &\quad \lesssim \int_{u_m}^u \int_{S_{t,u'}} (|\underline{\pi}|^2 (|\hat{\chi}|^2 + |\zeta|^2) + |\hat{\chi}| \cdot |\underline{\pi}|^3) d\mu_\gamma du' \\ &\quad \le C(u-u_m)\end{aligned}$$

Combining all the above estimates with (8.61) and noting that $u - u_m \lesssim \tau \lesssim 1$, it yields

$$\|\mathrm{tr}\chi\underline{\pi}\|^2_{L^2(\mathrm{Int}(S_{t,u}))} \le C + C\|\mathrm{tr}\chi\underline{\pi}\|_{L^2(\mathrm{Int}(S_{t,u}))}$$

which implies $\|tr\chi\underline{\pi}\|_{L^2(\mathrm{Int}(S_{t,u}))} \le C$. This together with (**BA2**) implies the desired inequality. □

9. **Proof of Theorem 4.7**

In this section we will complete the proof of Theorem 4.7. For any $p \in \mathcal{M}_I$, let $\Phi(t)$ be the integral curve of $\mathbf{T}$ through p with $\Phi(t(p)) = p$. For each $p_t := \Phi(t)$, we will represent $k(p_t)$ in terms of a Kirchoff-Sobolev formula over a past null cone with vertex p_t. We then use the estimates established in the previous section to obtain $\int_{t(p)-\tau}^{t(p)} |k(\Phi(t))|^2 n dt \le C$ for some universal constant C.

9.1. **Derivation of Kirchoff Parametrix.** We first revisit the formulation of Kirchoff Parametrix in [10]. We define $\mathbf{A}$ to be a Σ_t tangent 2-tensor verifying

$$(\mathbf{D}_L \mathbf{A})_{ij} + \frac{1}{2}\mathrm{tr}\chi \mathbf{A}_{ij} = 0 \quad \text{on } \mathcal{N}^-(p,\tau), \qquad \lim_{t\to t(p)^-}(t(p)-t)\mathbf{A}_{ij} = J_{ij}, \tag{9.1}$$

where $J \in T_p\Sigma_{t(p)}$ and $|J|_g = 1$. This $\mathbf{A}$ is similar to the one defined in [12] but with the modification that $\mathbf{A}$ is Σ_t tangent. Since we have obtained in Propositions 13 and 14 the estimates on

$$\left\|\mathrm{tr}\chi - \frac{2}{s}\right\|_{L^\infty}, \ \|\nabla\!\!\!/\,\mathrm{tr}\chi\|_{L^2}, \ \|r^{\frac{1}{2}}\nabla\!\!\!/\,\mathrm{tr}\chi\|_{L_x^2 L_t^\infty}, \ \|r^{-1}(\zeta+\underline{\zeta})\|_{L^2}, \ \|\hat{\chi}, \nu, \underline{\zeta}\|_{L_\omega^\infty L_t^2}, \ \mathcal{R}(p,\tau)$$

on the null cone $\mathcal{N}^-(p,\tau)$, we may adapt the proof in [12] to obtain the following estimates on $\mathbf{A}$.

Proposition 16. *For the tensor* $\mathbf{A}$ *defined by (9.1) there hold*

$$\|\nabla\!\!\!/ \boldsymbol{A}\|_{L^2(\mathcal{N}^-(p,\tau))} + \|r^{\frac{1}{2}}\nabla\!\!\!/ \boldsymbol{A}\|_{L_x^2 L_t^\infty(\mathcal{N}^-(p,\tau))} + \|r\boldsymbol{A}\|_{L^\infty(\mathcal{N}^-(p,\tau))} \le C, \tag{9.2}$$

where C *is a universal constant.*

Now we revisit the Kirchoff-Sobolev formula for any Σ_t tangent 2-tensor Ψ_I, $I = \{\mu,\nu\}$, see [10, 15]. According to the definition of $\Box\Psi_I$, we have under the null frame $(e_A)_{A=1,2}$, $e_3 = \underline{L}$, $e_4 = L$ that

$$\Box\Psi_I = -\frac{1}{2}\mathbf{D}_{43}\Psi_I - \frac{1}{2}\mathbf{D}_{34}\Psi_I + \delta^{AB}\mathbf{D}_{AB}\Psi_I.$$

By (8.1),

$$\mathbf{D}_{43}\Psi_I = \mathbf{D}_4(\mathbf{D}_3\Psi)_I - 2\underline{\zeta}^A \mathbf{D}_A\Psi_I. \tag{9.3}$$

It is easy to see

$$\mathbf{D}_{34}\Psi_I - \mathbf{D}_{43}\Psi_I = \mathbf{R}_\mu{}^\alpha{}_{34}\Psi_{\alpha\nu} + \mathbf{R}_\nu{}^\alpha{}_{34}\Psi_{\mu\alpha}.$$

By (8.2), we obtain

$$\delta^{AB}\mathbf{D}_{AB}\Psi_I = \delta^{AB}\nabla\!\!\!/_A\nabla\!\!\!/_B\Psi_I - \frac{1}{2}\mathrm{tr}\underline{\chi}\mathbf{D}_4\Psi_I - \frac{1}{2}\mathrm{tr}\chi\mathbf{D}_3\Psi_I.$$

Therefore

$$\begin{aligned}\Box\Psi_I = &-\mathbf{D}_4(\mathbf{D}_3\Psi)_I + 2\underline{\zeta}^A\mathbf{D}_A\Psi_I - \frac{1}{2}\mathrm{tr}\underline{\chi}\mathbf{D}_4\Psi_I - \frac{1}{2}\mathrm{tr}\chi\mathbf{D}_3\Psi_I \\ &+ \delta^{AB}\nabla\!\!\!/_A\nabla\!\!\!/_B\Psi_I - \frac{1}{2}\mathbf{R}_\mu{}^\alpha{}_{34}\Psi_{\alpha\nu} - \frac{1}{2}\mathbf{R}_\nu{}^\alpha{}_{34}\Psi_{\mu\alpha}.\end{aligned}$$

We multiply the above equation by $\mathbf{A}_I$ and integrate over $\mathcal{N}^-(p,\tau)$ to obtain

$$\begin{aligned}\int_{\mathcal{N}^-(p,\tau)} \Box\Psi_I \mathbf{A}^I = \Xi_1 + \Xi_2 + &\int_{\mathcal{N}^-(p,\tau)} \left(2\underline{\zeta}^A \mathbf{D}_A \Psi_I \cdot \mathbf{A}^I + \delta^{AB} \nabla\!\!\!\!/_A \nabla\!\!\!\!/_B \Psi_I \cdot \mathbf{A}^I\right)\\ &- \frac{1}{2}\int_{\mathcal{N}^-(p,\tau)} \left(\mathbf{R}_\mu{}^\alpha{}_{34}\Psi_{\alpha\nu} + \mathbf{R}_\nu{}^\alpha{}_{34}\Psi_{\mu\alpha}\right)\mathbf{A}^{\mu\nu}. \end{aligned}\tag{9.4}$$

where

$$\begin{aligned}\Xi_1 &= \int_{\mathcal{N}^-(p,\tau)} \left(-\mathbf{D}_4(\mathbf{D}_3\Psi)_I \cdot \mathbf{A}^I - \frac{1}{2}\mathrm{tr}\chi \mathbf{D}_3\Psi_I \cdot \mathbf{A}^I\right),\\ \Xi_2 &= -\frac{1}{2}\int_{\mathcal{N}^-(p,\tau)} \mathrm{tr}\underline{\chi}\mathbf{D}_4\Psi_I \cdot \mathbf{A}^I.\end{aligned}$$

For Ξ_1, integrating by parts gives

$$\begin{aligned}\Xi_1 &= \int_{\mathcal{N}^-(p,\tau)} \left(-\mathbf{D}_4(\mathbf{D}_3\Psi)_I \cdot \mathbf{A}^I - \mathrm{tr}\chi\mathbf{D}_3\Psi_I\cdot\mathbf{A}^I + \frac{1}{2}\mathrm{tr}\chi \mathbf{D}_3\Psi_I \cdot \mathbf{A}^I\right)\\ &= -\int_{S_{t(p)-\tau}} \mathbf{D}_3\Psi_I\cdot\mathbf{A}^I + \lim_{t\to t(p)}\int_{S_t}\mathbf{D}_3\Psi_I\cdot\mathbf{A}^I\\ &\quad + \int_{\mathcal{N}^-(p,\tau)}\left(\mathbf{D}_4\mathbf{A}^I + \frac{1}{2}\mathrm{tr}\chi\mathbf{A}^I\right)\cdot\mathbf{D}_3\Psi_I\end{aligned}$$

Since $\lim_{t\to t(p)}(t(p)-t)^2\mathbf{A} = 0$, we have in view of (9.1) that

$$\Xi_1 = -\int_{S_{t(p)-\tau}}\mathbf{D}_3\Psi_I\cdot\mathbf{A}^I + \int_{\mathcal{N}^-(p,\tau)}\Omega_1(\Psi),$$

where

$$\Omega_1(\Psi) = \mathbf{D}_4\mathbf{A}^{0i}\cdot\mathbf{D}_3\Psi_{0i} + \mathbf{D}_4\mathbf{A}^{i0}\cdot\mathbf{D}_3\Psi_{i0}.$$

For the term Ξ_2, in view of (9.1) and the fact that Ψ is Σ_t tangent, we first have

$$\begin{aligned}-\frac{1}{2}\mathrm{tr}\underline{\chi}\mathbf{D}_4\Psi_I\cdot\mathbf{A}^I &= -\frac{1}{2}\left(\mathbf{D}_4(\Psi_I\cdot\mathbf{A}^I\mathrm{tr}\underline{\chi}) - \mathbf{D}_4\mathbf{A}^I\cdot\mathrm{tr}\underline{\chi}\cdot\Psi_I - \mathbf{D}_4\mathrm{tr}\underline{\chi}\cdot\Psi_I\cdot\mathbf{A}^I\right)\\ &= -\frac{1}{2}\left(\mathbf{D}_4(\Psi_I\cdot\mathbf{A}^I\mathrm{tr}\underline{\chi}) + \frac{1}{2}\mathrm{tr}\chi\mathrm{tr}\underline{\chi}\mathbf{A}^I\cdot\Psi_I - \mathbf{D}_4\mathrm{tr}\underline{\chi}\cdot\Psi_I\cdot\mathbf{A}^I\right),\end{aligned}$$

thus integration by parts yields

$$\Xi_2 = \int_{\mathcal{N}^-(p,\tau)}\frac{1}{2}\underline{\mu}\mathbf{A}^I\cdot\Psi_I - \frac{1}{2}\left(\int_{S_{t(p)-\tau}}\Psi_I\cdot\mathbf{A}^I\mathrm{tr}\underline{\chi} - \lim_{t\to t(p)}\int_{S_t}\Psi_I\cdot\mathbf{A}^I\mathrm{tr}\underline{\chi}\right),$$

where $\underline{\mu}$ is defined in (8.11).

In view of $\mathrm{tr}\underline{\chi}' = -\mathrm{tr}\chi' - 2\delta^{AB}k_{AB}$, we have

$$\lim_{t\to t(p)}\frac{1}{2}\int_{S_t}\Psi_I\cdot\mathbf{A}^I\mathrm{tr}\underline{\chi} = -4\pi n(p)\langle\Psi, J\rangle,$$

Hence

$$\Xi_2 = \int_{\mathcal{N}^-(p,\tau)}\frac{1}{2}\underline{\mu}\mathbf{A}^I\cdot\Psi_I - \frac{1}{2}\int_{S_{t(p)-\tau}}\Psi_I\cdot\mathbf{A}^I\mathrm{tr}\underline{\chi} - 4\pi n(p)\langle\Psi, J\rangle.$$

Therefore we derive

$$
\begin{aligned}
4\pi n(p)\langle\Psi, J\rangle = & \int_{\mathcal{N}^-(p,\tau)} \left(-\Box\Psi_I\cdot\mathbf{A}^I+\frac{1}{2}\underline{\mu}\Psi_I\cdot\mathbf{A}^I+\Omega_1(\Psi)\right)\\
&-\int_{S_{t(p)-\tau}}\left(\mathbf{D}_3\Psi_I\cdot\mathbf{A}^I+\frac{1}{2}\mathrm{tr}\underline{\chi}\Psi_I\cdot\mathbf{A}^I\right)\\
&+\int_{\mathcal{N}^-(p,\tau)}\left(2\underline{\zeta}^B\mathbf{D}_B\Psi_I\cdot\mathbf{A}^I-\nabla\!\!\!\!/_B\Psi_I\cdot\nabla\!\!\!\!/^B\mathbf{A}^I\right)\\
&-\frac{1}{2}\int_{\mathcal{N}^-(p,\tau)}\left(\mathbf{R}_i{}^\alpha{}_{34}\Psi_{\alpha j}+\mathbf{R}_j{}^\alpha{}_{34}\Psi_{i\alpha}\right)\mathbf{A}^{ij}.
\end{aligned} \tag{9.5}
$$

We apply (9.5) to the tensor field $\Psi = k$ and obtain

Theorem 9.1. *Let $p\in\mathcal{M}_I$, let $\Phi(t)$ be the integral curve of $\mathbf{T}$ through p with $\Phi(t(p)) = p$, and let $p_t = \Phi(t)$. Let $\mathbf{A}$ be a Σ_t tangent 2-tensor on $\mathcal{J}^-(p,\tau)$ verifying (9.1) on each null cone $C_u := \mathcal{N}^-(p_t, t-t(p)+\tau)$, where $u = u(t) = \int_{t_0}^t n|_\Phi dt$ for $t_m := t(p)-\tau\le t\le t(p)$. Then there holds*

$$
4\pi n(p_t)\langle k(p_t), J\rangle = I(p_t)+J(p_t)+K(p_t)+L(p_t)+\mathfrak{E}(p_t)+\int_{C_u}\Omega_1(k), \tag{9.6}
$$

where

$$
\begin{aligned}
I(p_t) &= -\int_{C_u}\mathbf{A}\cdot\Box k,\\
J(p_t) &= -\frac{1}{2}\int_{C_u}\mathbf{A}\cdot\mathbf{R}(\cdot,\cdot,\underline{L},L)\cdot k,\\
K(p_t) &= \int_{C_u}\left(-\nabla\!\!\!\!/^B\mathbf{A}\cdot\nabla\!\!\!\!/_B k+2\underline{\zeta}^B\cdot\nabla\!\!\!\!/_B k\cdot\mathbf{A}\right),\\
L(p_t) &= \frac{1}{2}\int_{C_u}\underline{\mu}\mathbf{A}\cdot k,\\
\mathfrak{E}(p_t) &= -\int_{S_{t_m,u}}\left(\mathbf{D}_3 k\cdot\mathbf{A}+\frac{1}{2}tr\underline{\chi}k\cdot\mathbf{A}\right).
\end{aligned}
$$

9.2. **Main estimates.** In the following we will use the representation formula given in Theorem 9.1 to show that

$$
\int_{t(p)-\tau}^{t(p)}|k(p_t)|^2 n dt\le C
$$

for some universal constant C. We proceed as follows.

• *Estimate on $I(p_t)$:* We will use the expression of $\Box k$ given in Proposition 8, which symbolically can be written as

$$
\Box k = -n^{-3}\dot{n}\nabla^2 n+n^{-2}\nabla^2\dot{n}+\pi\cdot\pi\cdot\pi+k\cdot\nabla^2 n+k\cdot Ric+\pi\cdot\nabla k-n^{-1}k.
$$

It then follows from Proposition 16 that

$$\begin{aligned}|I(p_t)| &\lesssim \int_{C_u} r^{-1}\left(|\dot n\nabla^2 n| + |\nabla^2\dot n| + |\pi|^3 + |k||\nabla^2 n| + |k||Ric| + |\pi||\nabla k| + |k|\right)\\ &\lesssim \|\nabla^2 n\|_{L^2(C_u)}\|r^{-1}\dot n\|_{L^2(C_u)} + \|r^{-1}\nabla^2\dot n\|_{L^1(C_u)} + \int_{C_u} r^{-1}|\pi|^3\\ &\quad + \|r^{-1}k\|_{L^2(C_u)}\|\nabla^2 n\|_{L^2(C_u)} + \|Ric\|_{L^2(C_u)}\|r^{-1}k\|_{L^2(C_u)}\\ &\quad + \|r^{-1}\pi\|_{L^2(C_u)}\|\nabla k\|_{L^2(C_u)} + \|r^{-1}k\|_{L^1(C_u)}.\end{aligned}$$

Therefore, with the help of Proposition 11 and Proposition 12, we have

$$\begin{aligned}|I(p_t)| \lesssim \|r^{-1}\dot n\|_{L^2(C_u)} + \|r^{-1}\nabla^2\dot n\|_{L^1(C_u)} + \int_{C_u} r^{-1}|\pi|^3\\ + \|Ric\|_{L^2(C_u)} + \|\nabla k\|_{L^2(C_u)} + C.\end{aligned}$$

Now we consider $\int_{t_m}^{t(p)} |I(p_t)|^2 dt$. Using $\frac{du}{dt} = n$, we have from Proposition 15 that

$$\begin{aligned}\int_{t_m}^{t(p)} \|r^{-1}\dot n\|^2_{L^2(C_{u(t)})} n dt &= \int_{u(t_m)}^{u(t(p))} \|r^{-1}\dot n\|^2_{L^2(C_u)} du\\ &= \int_{u(t_m)}^{u(t(p))}\int_{t_m}^{t_M(u)}\int_{S_{t',u}} r^{-2}|\dot n|^2 n a d\mu_\gamma dt' du\\ &= \int_{t_m}^{t(p)}\int_{u(t')}^{u(t(p))}\int_{S_{t',u}} r^{-2}|\dot n|^2 n a d\mu_\gamma du dt'\\ &\lesssim \int_{t_m}^{t(p)} \|r^{-1}\dot n\|^2_{L^2(Int(S_{t',u(t(p))}))} dt'\\ &\le C\tau.\end{aligned}$$

By similar argument, we have from Lemma 2.2 that

$$\int_{t_m}^{t(p)} \left(\|Ric\|^2_{L^2(C_u)} + \|\nabla k\|^2_{L^2(C_u)}\right) n dt \le C\tau.$$

Therefore

$$\int_{t_m}^{t(p)} |I(p_t)|^2 n dt \lesssim C\tau + \int_{t_m}^{t(p)} \|r^{-1}\nabla^2\dot n\|^2_{L^1(C_u)} n dt + \int_{t_m}^{t(p)} \left(\int_{C_u} r^{-1}|\pi|^3\right)^2 n dt.$$

By using the Minkowski inequality and Proposition 7 we have

$$\begin{aligned}&\left(\int_{t_m}^{t(p)} \|r^{-1}\nabla^2\dot n\|^2_{L^1(C_u)} n dt\right)^{1/2}\\ &\quad= \left(\int_{u(t_m)}^{u(t(p))}\left(\int_{t_m}^{t_M(u)} r^{-1}\|an\nabla^2\dot n\|_{L^1(S_{t',u})} dt'\right)^2 du\right)^{1/2}\\ &\quad\le \int_{t_m}^{t(p)}\left(\int_{u(t')}^{u(t(p))} r^{-2}\|an\nabla^2\dot n\|^2_{L^1(S_{t',u})} du\right)^{1/2} dt'\\ &\quad\lesssim \int_{t_m}^{t(p)} \|\nabla^2\dot n\|_{L^2(Int(S_{t',u(t(p))}))} dt' \le C.\end{aligned}$$

Finally, we have from Proposition 11 and (8.22) that

$$\int_{C_u} r^{-1}|\pi|^3 \lesssim \int_{t_m}^{t_M(u)} \|r^{-1}\pi\|_{L^2(S_{t',u})}\|\pi\|^2_{L^4(S_{t',u})}dt' \le C(t_M(u)-t_m)^{1/2}.$$

Thus, by Lemma 7.1 we obtain

$$\int_{t_m}^{t(p)} \left(\int_{C_u} r^{-1}|\pi|^3\right)^2 ndt \le C\tau^2.$$

Combining the above estimates we therefore obtain

$$\int_{t_m}^{t(p)} |I(p_t)|^2 ndt \le C + C\tau^2 \lesssim C.$$

- *Estimate on $J(p_t)$:* It follows from Proposition 16, Theorem 4.4 and Proposition 12 that

$$|J(p_t)| \lesssim \|r\mathbf{A}\|_{L^\infty(C_u)}\|r^{-1}k\|_{L^2(C_u)}\mathcal{R}(p_t, \tau+t-t(p)) \le C.$$

Thus

$$\int_{t_m}^{t(p)} |J(p_t)|^2 ndt \le C(t(p)-t_m) \le C\tau \le C.$$

- *Estimate on $K(p_t)$:* It follows from the Hölder inequality that

$$|K(p_t)| \lesssim \|\nabla\!\!\!/\mathbf{A}\|_{L^2(C_u)}\|\nabla\!\!\!/ k\|_{L^2(C_u)} + \|r\mathbf{A}\|_{L^\infty(C_u)}\|r^{-1}\underline{\zeta}\|_{L^2(C_u)}\|\nabla\!\!\!/ k\|_{L^2(C_u)}.$$

Thus, we obtain from Proposition 16, Theorem 4.4, and Proposition 12 that $|K(p_t)| \le C$ which gives

$$\int_{t_m}^{t(p)} |K(p_t)|^2 ndt \le C(t(p)-t_m) \le C\tau \le C.$$

- *Estimate on $L(p_t)$:* It follows from Proposition 16 and Proposition 12 that

$$|L(p_t)| \le \|r\mathbf{A}\|_{L^\infty(C_u)}\|r^{-1}\bar{\pi}\|_{L^2(C_u)}\|\underline{\mu}\|_{L^2(C_u)} \lesssim \|\underline{\mu}\|_{L^2(C_u)}.$$

From Lemma 8.5 we then obtain $|L(p_t)| \le C$. Therefore

$$\int_{t_m}^{t(p)} |L(p_t)|^2 ndt \le C(t(p)-t_m) \le C\tau \le C.$$

- *Estimate on $\mathfrak{E}(p_t)$:* We first have from Proposition 16 that

$$|\mathfrak{E}(p_t)| \lesssim r^{-1}\|\mathbf{D}_3 k\|_{L^1(S_{t_m,u})} + r^{-1}\|\mathrm{tr}\underline{\chi} k\|_{L^1(S_{t_m,u})}.$$

Using the definition of r we then obtain

$$|\mathfrak{E}(p_t)| \lesssim \|\mathbf{D}_3 k\|_{L^2(S_{t_m,u})} + r^{-1}\|\mathrm{tr}\underline{\chi} k\|_{L^1(S_{t_m,u})}.$$

Recall

$$\mathrm{tr}\underline{\chi}' = -\mathrm{tr}\chi' - 2\delta^{AB}k_{AB}.$$

Since (**BA1**) implies $1/2 \le a \le 3/2$. Thus, with the help of (**BA2**), it yields

$$\begin{aligned}\|\mathrm{tr}\underline{\chi} k\|_{L^1(S_{t_m,u})} &\lesssim \left\|\mathrm{tr}\chi - \frac{2}{s}\right\|_{L^\infty(C_u)}\|k\|_{L^1(S_{t_m,u})} + r^{-1}\|k\|_{L^1(S_{t_m,u})} + \|k\|^2_{L^2(S_{t_m,u})}\\ &\lesssim r^{-1}\|k\|_{L^1(S_{t_m,u})} + \|k\|^2_{L^2(S_{t_m,u})}\\ &\lesssim \|k\|_{L^2(S_{t_m,u})} + r\|k\|^2_{L^4(S_{t_m,u})}.\end{aligned}$$

Consequently

$$|\mathfrak{E}(p_t)| \lesssim \|\mathbf{D}_3 k\|_{L^2(S_{t_m,u})} + r^{-1}\|k\|_{L^2(S_{t_m,u})} + \|k\|^2_{L^4(S_{t_m,u})}.$$

Therefore, using $\frac{du}{dt} = n$, we have

$$\begin{aligned}\int_{t_m}^{t(p)} |\mathfrak{E}(p_t)|^2 dt &\lesssim \int_{u(t_m)}^{u(t(p))} |\mathfrak{E}(p_t)|^2 du \\ &\lesssim \|\mathbf{D}_3 k\|^2_{L^2(\Sigma_{t_m})} + \|r^{-1}k\|^2_{L^2(Int(S_{t_m,u}))} + \|k\|^4_{L^4(\Sigma_{t_m})}\end{aligned}$$

It follows from Lemma 2.2 and Proposition 15 that

$$\int_{t_m}^{t(p)} |\mathfrak{E}(p_t)|^2 dt \lesssim \|\mathbf{D}_3 k\|^2_{L^2(\Sigma_{t_m})} + C.$$

Recall that $\underline{L} = -a(\mathbf{T} - N)$. So $\mathbf{D}_3 k = -a(\mathbf{D}_0 k - \nabla_N k)$. Recall also that $\mathbf{D}_0 k = -n^{-1}\nabla^2 n + Ric + k\mathrm{Tr}k$. Thus

$$\|\mathbf{D}_3 k\|_{L^2(\Sigma_{t_m})} \lesssim \|\nabla^2 n\|_{L^2(\Sigma_{t_m})} + \|Ric\|_{L^2(\Sigma_{t_m})} + \|k\|^2_{L^4(\Sigma_{t_m})} + \|\nabla k\|_{L^2(\Sigma_{t_m})}.$$

It follows from Lemma 2.2 and Proposition 3 that $\|\mathbf{D}_3 k\|_{L^2(\Sigma_{t_m})} \le C$. Therefore

$$\int_{t_m}^{t(p)} |\mathfrak{E}(p_t)|^2 n dt \le C.$$

• *Estimate on* $\int_{C_u} \Omega_1(k)$*:* By straightforward calculation we have $\Omega_1(k) = \mathbf{A} \cdot \bar{\pi} \cdot \bar{\pi} \cdot \bar{\pi}$. It follows from Proposition 16 that

$$|\Omega_1(k)| \lesssim \int_{C_u} r^{-1}|\bar{\pi}|^3.$$

Therefore, one can use the similar argument in the estimate of $I(p_t)$ to get

$$\int_{t_m}^{t(p)} |\Omega_1(k)|^2 n dt \lesssim \int_{t_m}^{t(p)} \left|\int_{C_u} r^{-1}|\bar{\pi}|^3\right|^2 n dt \le C\tau^2 \le C.$$

10. Proof of main theorem I

In this section, based on Theorem 1.2, we will follow the idea in [12] to give the proof of Theorem 1.1. According to the local existence theorem given in [12, Proposition 6.1], see also [5, Theorem 10.2.1], it suffices to show that the quantity

$$\mathcal{R}_* := \|Ric\|_{H^2(\Sigma_t)} + \|k\|_{H^3(\Sigma_t)} \tag{10.1}$$

on each slice Σ_t with $t_0 \le t < t_*$ is uniformly bounded.

Since $(\mathbf{M}, \mathbf{g})$ is a vacuum space-time, by virtue of the Bianchi identity $\mathbf{R}$ verifies a wave equation of the form

$$\Box \mathbf{R} = \mathbf{R} \star \mathbf{R}, \tag{10.2}$$

Based on higher energy estimates it is standard to show that

$$\|\mathbf{D}\mathbf{R}(t)\|^2_{L^2} \lesssim \|\mathbf{D}\mathbf{R}(t_1)\|^2_{L^2} + \int_{t_1}^t \|\mathbf{R}(t')\|^2_{L^\infty} dt' \tag{10.3}$$

and

$$\|\mathbf{D}^2\mathbf{R}(t)\|^2_{L^2} \lesssim \|\mathbf{D}^2\mathbf{R}(t_1)\|^2_{L^2} + \int_{t_1}^t \|\mathbf{D}\mathbf{R}(t')\|^2_{L^2}\|\mathbf{R}(t')\|^2_{L^\infty} dt' \tag{10.4}$$

for all $t_0 \leq t_1 \leq t < t_*$. The derivation has been given in [12] under the assumption (1.9), the argument however depends only on the condition (**A1**).

Thus, the derivation of the L^∞ bound of $\mathbf{R}$ is a crucial step. As in [10] one can represent $\mathbf{R}(p)$, for each $p \in \mathcal{M}_*$, by a Kirchoff-Sobolev formula over the null cone $\mathcal{N}^-(p,\tau)$, where $\tau > 0$ is a universal constant such that $i_*(p,t) \geq \tau$ whose existence is guaranteed by Theorem 1.2. One can then follow the delicate argument in [12] to derive that

$$\|\mathbf{R}(t)\|_{L^\infty} \lesssim \tau^{-1} \sup_{t' \in [t-2\tau, t-\tau/2]} \left(\|\mathbf{R}(t')\|_{L^2} + \|\mathbf{D}\mathbf{R}(t')\|_{L^2} + \|\mathbf{D}^2\mathbf{R}(t')\|_{L^2} \right). \tag{10.5}$$

The derivation of (10.5) requires the estimates on

$$\mathcal{R}(p,\tau), \quad \left\| \mathrm{tr}\chi - \frac{2}{s} \right\|_{L^\infty(\mathcal{N}^-(p,\tau))}, \quad \|\hat{\chi}, \nu, \zeta, \underline{\zeta}\|_{L^\infty_\omega L^2_t(\mathcal{N}^-(p,\tau))},$$
$$\|\mu, \nabla\!\!\!/ \mathrm{tr}\chi\|_{L^2(\mathcal{N}^-(p,\tau))}, \quad \|r^{1/2} \nabla\!\!\!/ \mathrm{tr}\chi\|_{L^2_x L^\infty_t(\mathcal{N}^-(p,\tau))}, \quad \|r^{-1}(\zeta + \underline{\zeta})\|_{L^2(\mathcal{N}^-(p,\tau))}$$

which are provided by Proposition 13 and Proposition 14 under the condition (**A1**). Combining the estimates (10.3)–(10.5) gives

$$\|\mathbf{R}(t)\|_{H^2} \lesssim \tau^{-1} \sup_{t' \in [t-\tau, t-\tau/2]} \|\mathbf{R}(t')\|_{H^2}.$$

Iterating this estimate as many times as needed, in steps of size $\tau/2$, yields

$$\sup_{t \in [t_0, t_*)} \|\mathbf{R}(t)\|_{H^2} \leq C, \tag{10.6}$$

where C is a positive constant depending only on Q_0, $\mathcal{K}_0$, $|\Sigma_0|$, t_*, I_0 and the initial data $\|\mathbf{R}(t_0)\|_{H^2}$.

Now we are ready to show that the quantity $\mathcal{R}_*$ defined by (10.1) is uniformly bounded for all $t_0 \leq t < t_*$. Although the argument is standard, we will include the details for completeness.

We have defined in (2.5) the electric and magnetic parts E, H of the curvature tensor $\mathbf{R}$. It is known that

$$\nabla_i k_{jm} - \nabla_j k_{im} = \epsilon_{ij}{}^l H_{lm}, \tag{10.7}$$

$$R_{ij} - k_{ia} k^{aj} + \mathrm{Tr} k \, k_{ij} = E_{ij}. \tag{10.8}$$

From Lemma 2.1 and Lemma 2.2 it follows that

$$\|Ric\|_{L^2} + \|k\|_{H^1} + \|E\|_{L^2} + \|H\|_{L^2} \leq C, \tag{10.9}$$

where and in the following all the norms are taken over a fixed slice Σ_t which is suppressed for simplicity.

In order to obtain the derivative estimates, by straightforward calculation we have symbolically

$$\nabla_m E_{ij} = \mathbf{D}_m \mathbf{R}_{0i0j} - k \cdot H, \tag{10.10}$$

$$\nabla_m H_{ij} = \mathbf{D}_m {}^\star\mathbf{R}_{0i0j} - k \cdot E, \tag{10.11}$$

$$\nabla^2_{mn} E_{ij} = \mathbf{D}^2_{mn} \mathbf{R}_{0i0j} - k_{mn} \mathbf{D}_0 \mathbf{R}_{0i0j} - \nabla(k \cdot H), \tag{10.12}$$

$$\nabla^2_{mn} H_{ij} = \mathbf{D}^2_{mn} {}^\star\mathbf{R}_{0i0j} - k_{mn} \mathbf{D}_0 {}^\star\mathbf{R}_{0i0j} - \nabla(k \cdot E). \tag{10.13}$$

From (10.10) and (10.11) it follows that

$$\|\nabla E\|_{L^2} \le \|\mathbf{DR}\|_{L^2} + \|k\|_{L^6}\|H\|_{L^3}$$
$$\|\nabla H\|_{L^2} \le \|\mathbf{DR}\|_{L^2} + \|k\|_{L^6}\|E\|_{L^3}$$

Applying Lemma 2.3 to $\|E\|_{L^3}$ and $\|H\|_{L^3}$, and using (10.6) and (10.9), we obtain

$$\|\nabla E\|_{L^2} + \|\nabla H\|_{L^2} \le C + C\left(\|\nabla E\|_{L^2}^{1/2} + \|\nabla H\|_{L^2}^{1/2}\right)$$

which implies

$$\|\nabla E\|_{L^2} + \|\nabla H\|_{L^2} \le C. \tag{10.14}$$

Next we will derive the estimate for $\|\nabla^2 k\|_{L^2}$. It follows from $\mathrm{div} k = 0$ and (10.7) that $\triangle k = Ric \cdot k + \nabla H$. Differentiating it and commuting ∇ with Δ gives

$$\triangle \nabla k = Ric \cdot \nabla k + \nabla Ric \cdot k + \nabla^2 H$$

which together with (10.8) implies

$$\triangle \nabla k = k \cdot k \cdot \nabla k + E \cdot \nabla k + \nabla E \cdot k + \nabla^2 H. \tag{10.15}$$

Multiplying (10.15) by ∇k and integrating over Σ_t yields

$$\begin{aligned}
\int_{\Sigma_t} |\nabla^2 k|^2 &\lesssim \int_{\Sigma_t} \left(|k|^2|\nabla k|^2 + |E||\nabla k|^2 + |\nabla E||k||\nabla k| + |\nabla H||\nabla^2 k|\right)\\
&\lesssim \|k\|_{L^6}^2\|\nabla k\|_{L^3}^2 + \|E\|_{L^6}\|\nabla k\|_{L^{12/5}}^2 + \|\nabla E\|_{L^2}\|\nabla k\|_{L^3}\|k\|_{L^6}\\
&\quad + \|\nabla H\|_{L^2}\|\nabla^2 k\|_{L^2}.
\end{aligned}$$

With the help of Lemma 2.3, (10.9) and (10.14), we have

$$\|\nabla^2 k\|_{L^2}^2 \le C\left(\|\nabla^2 k\|_{L^2} + \|\nabla^2 k\|_{L^2}^{1/2}\right)$$

which implies $\|\nabla^2 k\|_{L^2} \le C$. By the Sobolev embedding we obtain

$$\|k\|_{L^\infty} + \|k\|_{H^2} \le C. \tag{10.16}$$

Using (10.16) and (10.6), it follows easily from (10.8), (10.12) and (10.13) that

$$\|\nabla Ric\|_{L^2} + \|\nabla^2 Ric\|_{L^2} + \|\nabla^2 E\|_{L^2} + \|\nabla^2 H\|_{L^2} \le C.$$

Finally we derive the estimate on $\|\nabla^3 k\|_{L^2}$. By differentiating (10.15), commuting ∇ with $\triangle$ and using (10.8) we obtain

$$\triangle \nabla^2 k = k \cdot k \cdot \nabla^2 k + k \cdot \nabla k \cdot \nabla k + E \cdot \nabla^2 k + \nabla E \cdot \nabla k + \nabla^2 E \cdot k + \nabla^3 H.$$

Multiplying this equation by $\nabla^2 k$ and integrating over Σ_t it follows

$$\begin{aligned}
\int_{\Sigma_t} |\nabla^3 k|^2 &\lesssim \int_{\Sigma_t} \left(|k|^2|\nabla^2 k|^2 + |k||\nabla k|^2|\nabla^2 k| + |E||\nabla^2 k|^2 + |\nabla E||\nabla k||\nabla^2 k|\right)\\
&\quad + \int_{\Sigma_t} \left(|k||\nabla^2 E||\nabla^2 k| + |\nabla^2 H||\nabla^3 k|\right)\\
&\lesssim \|k\|_{L^\infty}^2\|\nabla^2 k\|_{L^2}^2 + \|k\|_{L^\infty}\|\nabla k\|_{L^4}^2\|\nabla^2 k\|_{L^2} + \|E\|_{L^\infty}\|\nabla^2 k\|_{L^2}^2\\
&\quad + \|\nabla E\|_{L^4}\|\nabla k\|_{L^4}\|\nabla^2 k\|_{L^2} + \|k\|_{L^\infty}\|\nabla^2 E\|_{L^2}\|\nabla^2 k\|_{L^2}\\
&\quad + \|\nabla^2 H\|_{L^2}\|\nabla^3 k\|_{L^2}\\
&\le C + C\|\nabla^3 k\|_{L^2}.
\end{aligned}$$

Therefore $\|\nabla^3 k\|_{L^2} \le C$. The proof is thus complete.

11. **Appendix**

In this appendix we give the proof of Lemma 7.3. It suffices to consider the case that F is an arbitrary smooth function on Σ_t.

On $\text{Int}S_{t,u_M(t)} = \cup_{u_m(t)\le u\le u_M(t)} S_{t,u}$, we have the family of diffeomorphisms

$$\Psi_{u,t} : \mathbb{S}^2 \to S_{t,u},\ \Psi_{u,t}(\omega) = \mathcal{G}_{\Phi(t_M(u))}(t,\omega),$$

where $\mathcal{G}$ is defined in (1.13). Relative to this radial foliation, the metric g on $\text{Int}S_{t,u_M}$ can be written as

$$a^2 du^2 + \gamma_{AB} d\omega_A d\omega_B,$$

where γ is the restriction of g on $S_{t,u}$. By Lemma 7.1, $\gamma_{S_{t,u}} \approx r(t,u)^2 \gamma_{\mathbb{S}^2}$. Moreover, $F(x)$, $x \in S_{t,u}$, can be reparametrized by

$$F(x) = F(u,\omega) := F\circ \Psi_{u,t}(\omega),\ \omega\in\mathbb{S}^2.$$

Due to Lemma 7.1 and (7.4), for a scalar function f,

$$\|f\|^2_{L^2(S_{t,u})} \approx \int_{\mathbb{S}^2} |f|^2 (u-u_m)^2 d\mu_{\mathbb{S}^2}. \tag{11.1}$$

For a fixed leaf S_{t,u_0} with $u_m(t) < u_0 \le u_M(t)$ and any $x \in S_{t,u_0}$, $F(x) = F(u_0,\omega)$ with $\omega\in\mathbb{S}^2$, we define with $u_m := u_m(t)$

$$\begin{aligned} m(x) &:= \frac{2}{u_0-u_m}\int_0^{\frac{u_0-u_m}{2}} F(-z+u_0,\omega)dz \\ G(x) &:= m(x) - F(x). \end{aligned}$$

Lemma 7.3 can be proved by establishing the following estimates

$$r^{-1/2}\|G\|_{L^2(S_{t,u_0})} \lesssim \|\nabla F\|_{L^2(\Sigma_t)} \tag{11.2}$$

$$r^{-1/2}\|m\|_{L^2(S_{t,u_0})} \lesssim \|F\|_{H^1(\Sigma_t)}, \tag{11.3}$$

where $r = r(t,u_0)$.

To see (11.2), according to definition, we have

$$G(x) = \frac{2}{u_0-u_m}\int_0^{\frac{u_0-u_m}{2}} \int_0^1 \frac{d}{d\ell} F(u_0-\ell z,\omega) d\ell dz. \tag{11.4}$$

It is easy to see

$$\frac{d}{d\ell}F(u_0-\ell z,\omega) = -z\cdot\partial_u F(u_0-\ell z,\omega).$$

In view of (**BA1**) and $N = -a^{-1}\partial_u$, it follows that

$$\left|\frac{d}{d\ell}F(u_0-\ell z,\omega)\right| \lesssim z|\nabla_N F|(u_0-\ell z,\omega). \tag{11.5}$$

Since $0 < z \le \frac{u_0-u_m}{2}$, we have from (7.4) that $z \lesssim r'$, where $r' = r(t, u_0-\ell z)$. Thus, by combining (11.4) with (11.5) and setting $v(y) := \|r'\nabla F(-y+u_0,\cdot)\|_{L^2_\omega}$ it

yields

$$\begin{aligned}
r^{-1/2}\|G\|_{L^2(S_{t,u_0})} &\lesssim r^{-1/2}\int_0^1 d\ell \int_0^{\frac{u_0-u_m}{2}} r' \||\nabla F(-\ell z+u_0,\cdot)|_g\|_{L^2_\omega} dz \\
&\lesssim r^{-1/2}\int_0^1 \ell^{-1}\int_0^{\frac{\ell(u_0-u_m)}{2}} v(y)dyd\ell \\
&\lesssim r^{-1/2}\int_0^{\frac{u_0-u_m}{2}}\int_{\frac{2y}{u_0-u_m}}^1 \ell^{-1}d\ell v(y)dy \\
&\lesssim (u_0-u_m)^{-\frac{1}{2}}\int_0^{\frac{u_0-u_m}{2}} \ln\left(\frac{u_0-u_m}{2y}\right)v(y)dy.
\end{aligned}$$

By Hölder inequality,

$$\begin{aligned}
r^{-1/2}\|G\|_{L^2(S_{t,u_0})} &\lesssim \left(\int_0^1 (\ln\sigma)^2 d\sigma\right)^{1/2}\left(\int_0^{\frac{u_0-u_m}{2}} |v(y)|^2 dy\right)^{1/2} \\
&\lesssim \|\nabla F\|_{L^2(\mathrm{Int}S_{t,u_0})}.
\end{aligned}$$

This proves (11.2). Using (7.4), with $r'' := r(t,u_0-z)\approx u_0-u_m-z$,

$$\|m\|_{L^2(S_{t,u_0})} \lesssim \int_0^{\frac{u_0-u_m}{2}} \|r''^{1/3}F\|_{L^6_\omega} r''^{-\frac{1}{3}}du' \lesssim r^{\frac{1}{2}}\|F\|_{L^6(\mathrm{Int}S_{t,u_0})}.$$

By Sobolev embedding, (11.3) follows.

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